An unfitted $hp$-adaptive finite element method based on hierarchical B-splines for interface problems of complex geometry

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Abstract

Generating finite element discretizations with direct interface parameterizations constitutes a considerable computational expense in case of complex interface geometries. The paper at hand introduces a B-spline finite element method, which circumvents parameterization of interfaces and offers fast and easy meshing irrespective of the geometric complexity involved. Its core idea is the adaptive approximation of discontinuities by hierarchical grid refinement, which adds several levels of local basis functions in the close vicinity of interfaces, but unfitted to their exact location, so that a simple regular grid of knot span elements can be maintained. Numerical experiments show that an $hp$-refinement strategy, which simultaneously increases the polynomial degree of the B-spline basis and the levels of refinement around interfaces, achieves exponential rates of convergence despite the presence of discontinuities. It is also demonstrated that the hierarchical B-spline FEM can be used to transfer the recently introduced Finite Cell concept to geometrically nonlinear problems. Its computational performance, imposition of unfitted boundary conditions and fast hierarchical grid generation are illustrated for a set of benchmark problems in one, two and three dimensions, and the advantages of the regular grid approach for complex geometries are demonstrated by the geometrically nonlinear simulation of a voxel based foam composite.

Keywords: High-order B-spline finite elements, Regular grid, Hierarchical refinement, Interface problems, Complex geometries, Finite Cell Method

1. Introduction

The key requirement for the mechanical simulation of multiphase materials is the reproduction of weak and strong discontinuities in stresses and strains, respectively, which appear along material interfaces as a result of the abrupt change of material parameters. A wide range of finite element approaches specifically tailored for interface problems is available today, comprising for example standard FEM with conforming mesh generation (Reid et al., 2009), generalized FEM (Strouboulis et al., 2001; Srinavasan et al., 2008) and eXtended FEM (Belytschko et al., 2001; Sukumar et al., 2001; Moës et al., 2003; Hetlich & Ramm, 2008), composite finite elements (Hackbusch & Sauter, 1997; Preusser et al., 2010), Lagrange multiplier, mortar and Nitsche based methods (Hansbo & Hansbo, 2002, 2004; Lamichhane & Wohlmuth, 2004; Mergheim & Steinmann, 2006; Flemisch & Wohlmuth, 2007; Dolbow & Harari, 2009) or $hp$-cloud (Duarte & Oden, 1996; Oden et al., 1998) and $hp$-$d$ approaches (Rank, 1992; Düster et al., 2007). However, most of these methods rely on an interface parameterization, e.g. in the form of a surface triangulation or level set function, which in the case of complex interface geometries lead to severe difficulties, since the parameterization process is hard to be automated, often requires time-consuming user interaction and is computationally very expensive. A considerable simplification in terms of the parameterization of complex domains has recently been achieved by the Finite Cell Method (FCM) (Parvizian et al., 2007; Düster et al., 2008). With the help of the fictitious domain concept, FCM allows for an easy regular grid of $p$-elements irrespective of the complexity of the domain, while the geometry

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information is implicitly taken into account during numerical integration. Similar ideas have been implemented in the immersed (Zhang et al., 2004) and unaligned finite element methods (Zohdi & Wriggers, 2005) and within the discontinuous Galerkin approach (Bastian & Engwer, 2009).

Besides isogeometric analysis, which has gained huge attention during the past few years (Hughes et al., 2005; Cottrell et al., 2009), uniform B-splines on regular grids have been demonstrated to constitute efficient finite element bases for arbitrary geometries (Höllig et al., 2002; Höllig, 2003; Embar et al., 2010; Sanches et al., 2011). In addition, the concept of hierarchical B-splines, originally developed for local surface refinement in CAD applications (Forsey & Bartels, 1988; Kraft, 1997), constitutes an effective way for local grid refinement in uniform B-spline bases, which has been successively applied for the adaptive resolution of singularities (Höllig, 2003). The present paper combines basic ideas from FCM and hierarchical B-splines for the construction of an unfitted \textit{hp}-adaptive finite element method, which is able to handle the discontinuities of interface problems without introducing a surface parameterization of their geometry. The finite element basis of the derived scheme consists of uniform B-splines on a regular grid, where arbitrary interface geometries are taken into account implicitly during numerical integration. In the vicinity of material interfaces, but not fitted to their exact locations, an adaptive grid refinement is realized by adding several levels of hierarchical B-splines. Similar to classical \textit{hp}-adaptivity (Šolín et al., 2004; Demkowicz, 2006; Demkowicz et al., 2007), the proposed method achieves exponential rates of convergence for discontinuous interface problems, when the polynomial degree of the B-spline basis and the number of hierarchical levels are simultaneously increased. The proposed hierarchical B-spline FEM shows strong similarities to the recently introduced \textit{hp}-d-adaptive Finite Cell Method, which combines a structured high-order mesh of \textit{p}-elements with an unfitted local refinement, using a hierarchical basis of linear hat functions (Schillinger et al., 2010). Interpreting a fictitious domain problem as a two-phase material with large difference in stiffness, hierarchical B-splines can also be applied in the sense of the Finite Cell Method. In contrast to the \textit{p}-version of the FCM, the hierarchical B-spline version of the FCM maintains high rates of convergence also for geometrically nonlinear fictitious domain problems, since oscillations around geometric boundaries can be effectively reduced by hierarchical refinement, without sacrificing the simple mesh generation property. However, the proposed refinement procedure does not achieve exponential rates of convergence in the FCM case. Note that a geometrically nonlinear formulation based on logarithmic strains is used throughout the paper in order to prevent the switching back and forth between different physical models.

The paper at hand is organized as follows: Section 2 introduces geometric nonlinearity in the framework of interface problems and briefly reviews classical nonlinear FE strategies. Section 3 describes the fundamentals of the hierarchical B-spline FEM in 1 dimension and illustrates accuracy and computational efficiency for a simple example. Section 4 outlines the oscillatory behavior of the \textit{p}-version FCM, which occurs as a result of a geometrically nonlinear formulation, and shows that the hierarchical B-spline version of the FCM can improve this issue. In Section 5, the hierarchical B-spline FEM is extended to 2 and 3 dimensions, where particular attention is devoted to the implementation of general boundary conditions. Section 6 introduces implicit and voxel based models for the representation of complex geometries as a basis for fast hierarchical grid generation. Finally, Section 7 presents a series of interface and fictitious domain examples in 2 and 3 dimensions.

2. Geometrically nonlinear formulation of interface problems

The geometrically nonlinear formulation of perfectly bonded interface problems based on the logarithmic strain measure is briefly reviewed in the following. Details can be found for instance in Belytschko et al. (2006), Bonet & Wood (2008), de Souza Neto et al. (2008) and Wriggers (2008).

2.1. Kinematics

In geometrically nonlinear statics, the deformation map \( \varphi \) describes the motion of each material particle from its undeformed reference configuration \( X \) to its deformed spatial configuration \( x \).

\[
x = \varphi(X)
\]

The displacement field \( u \) and the deformation gradient \( F \) follow as

\[
u = x - X
\]

2
In view of its polar decomposition, $F = VR$ can be represented in terms of the rotation tensor $R$ and the spatial stretch tensor $V$. Thus, $V^2$ can be obtained from the left Cauchy-Green tensor $b$ as

$$b = FF^T = (VR)(R^T V) = V^2$$

Evaluation of the principal directions of $V^2$ yields the orthogonal spatial Eigenvector triad $\{n_1, n_2, n_3\}$ and the associated Eigenvalues $\{\lambda_1^2, \lambda_2^2, \lambda_3^2\}$, which can be identified as the squared principal stretches. The spatial logarithmic strain tensor $\varepsilon = \ln V$ can then be computed in spectral form as

$$\varepsilon = \sum_{\alpha=1}^3 \ln \lambda_\alpha n_\alpha \otimes n_\alpha$$

Following Hansbo & Hansbo (2002) and Mergheim & Steinmann (2006), an elastic body consisting of $n$ different homogeneous material phases can be represented mathematically as a domain $\Omega$ with boundary $\partial \Omega$, which is divided into $n$ not necessarily compact open sets $\Omega_k$ by internal boundaries $\Gamma_{ij}$, denoting the material interfaces between $\Omega_i$ and $\Omega_j$ (see Fig. 1). For any sufficiently regular function $f$ in $\Omega_1 \cup \Omega_2 \cup ... \cup \Omega_n$, the jump of $f$ on $\Gamma_{ij}$ can be defined by

$$[f] := f_i|_{\Gamma_{ij}} - f_j|_{\Gamma_{ij}}$$

where $f_k = f|_{\Omega_k}$ is the restriction of $f$ to $\Omega_k$. Under the assumption of perfect bonding between the different phases, the deformation map $\varphi$ of Eq. (2.1) is sufficiently regular, and the following continuity conditions hold on each material interface $\Gamma_{ij}$

$$[\varphi] = \varphi_i - \varphi_j = 0$$

$$[F] = \frac{\partial \varphi_i}{\partial X} - \frac{\partial \varphi_j}{\partial X} = 0$$

Thus, the deformation map of a multiphase body and corresponding displacements Eqs. (2.1) and (2.2) generally exhibit a weak discontinuity (kink) along material interfaces, whereas the deformation gradient and related stretches and strains Eqs. (2.3) to (2.5) exhibit a strong discontinuity (jump).

2.2. Constitutive equations in principal directions

The Hencky hyperelastic model is the finite logarithmic strain-based extension of the standard linear elastic material, whose strain energy function $\Psi$ reads

$$\Psi = \frac{E_k}{2(1 + \nu_k)} \left( (\ln \lambda_1)^2 + (\ln \lambda_2)^2 + (\ln \lambda_3)^2 \right) + \nu_k \frac{E_k}{2(1 + \nu_k)(1 - 2\nu_k)} (\ln J)^2$$

with $J = \det F = \lambda_1 \lambda_2 \lambda_3$. Young’s modulus $E_k$ and Poisson’s ratio $\nu_k$ characterize elastic material of each sub-domain $k$. The principal Cauchy stresses along the principal axes $\alpha = \{1, 2, 3\}$ follow from Eq. (2.8) as

$$\sigma_\alpha = \frac{1}{J} \frac{\partial \Psi}{\partial \ln \lambda_\alpha} = \frac{E_k}{J(1 + \nu_k)} \ln \lambda_\alpha + \frac{\nu_k E_k}{J(1 + \nu_k)(1 - 2\nu_k)} (\ln J)$$
The fourth order spatial elasticity tensor in Cartesian coordinates can then be computed as
\[
c = \sum_{\alpha, \beta=1}^{3} \frac{1}{J} \frac{\partial^2 \Psi}{\partial \lambda_\alpha \partial \lambda_\beta} \eta_{\alpha \beta} \eta_{\alpha \beta} - \sum_{\alpha=1}^{3} 2 \sigma_\alpha \eta_{\alpha \alpha} + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \frac{\sigma_\alpha \lambda_\beta^2 - \sigma_\beta \lambda_\alpha^2}{\lambda_\alpha^2 - \lambda_\beta^2} (\eta_{\alpha \beta} \eta_{\alpha \beta})
\]  
(2.10)

implying the fourth order dyadic product \( \eta_{ijkl} = n_i \otimes n_j \otimes n_k \otimes n_l \). The evaluation of the Cartesian components \( c_{ijkl} \) of tensor \( c \) of Eq. (2.10) requires a transformation procedure described in detail for example in Bonet & Wood (2008). Due to the strong discontinuities in the stretches \( \lambda_\alpha \) of Eq. (2.8), all derived quantities exhibit jumps across the material interfaces.

2.3. Discretization, linearization and the Newton-Raphson procedure

On the basis of the principle of virtual work, the nonlinear equilibrium equations in the spatial configuration formulated in the framework of Eqs. (2.1) to (2.10) can be expressed in weak form as
\[
\delta W(\varphi, \delta \varphi) = \int_{\Omega} \sigma : \nabla \delta \varphi \, dv - \int_{\Omega} b \cdot \delta \varphi \, dv - \int_{\partial \Omega} t \cdot \delta \varphi \, da = 0
\]  
(2.11)

with body forces \( b \), traction vector \( t \) and virtual velocity \( \delta \varphi \). Together with suitable boundary conditions, Eq. (2.11) constitutes a boundary value problem, where integrals are evaluated in the spatial configuration. Kinematical quantities \( u, \delta \varphi \) and \( F \) can be discretized with a finite set of \( n \) shape functions \( N_a, a = 1, 2, ..., n \)
\[
u = \sum_{a=1}^{n} N_a u_a \]  
(2.12)
\[
\delta \varphi = \sum_{a=1}^{n} N_a \delta u_a \]  
(2.13)
\[
F = I + \sum_{a=1}^{n} \nabla X N_a u_a \]  
(2.14)

with corresponding discrete coefficients \( u_a \) and \( \delta u_a \). Inserting Eqs. (2.12) to (2.14) into Eq. (2.11), the discretized virtual work per coefficient \( a \) can be expressed as the difference between the internal and external equivalent force vectors \( f_{int}^a \) and \( f_{ext}^a \), called residual \( r_a \)
\[
\delta W(\varphi, N_a \delta u_a) = \delta \varphi_a \cdot (f_{int}^a - f_{ext}^a) = \delta \varphi_a \cdot r_a
\]  
(2.15)
\[
f_{int}^a = \int_{\Omega} B_a^T \sigma \, dv \]  
(2.16)
\[
f_{ext}^a = \int_{\Omega} N_a b \, dv + \int_{\partial \Omega} N_a t \, da
\]  
(2.17)

where \( B \) denotes the classical B-matrix. The consistent linearization of the discretized weak form Eq. (2.11) \( D \delta W \) in the direction of an incremental displacement \( \Delta u \) can be expressed in terms of material, geometric and loading parts \( D \delta W_c, D \delta W_\sigma \) and \( D \delta W_f \), respectively, as
\[
D \delta W(\varphi, N_a \delta u_a][N_b \Delta u_b] = D \delta W_c + D \delta W_\sigma - D \delta W_f
\]  
(2.18)
\[
D \delta W_c = \delta \varphi_a \cdot \left( \int_{\Omega} B_a^T \sigma \, dv \right) \Delta u_b
\]  
(2.19)
\[
D \delta W_\sigma = \delta \varphi_a \cdot \left( \int_{\Omega} \nabla X N_a \cdot \sigma \nabla X N_b \right) I \, dv \Delta u_b
\]  
(2.20)

where \( c \) is the array representation of the spatial elasticity tensor Eq. (2.10). In Eqs. (2.19) and (2.20), the expressions in brackets can be identified as the entries \( K_{c,ab} \) and \( K_{\sigma,ab} \) of the material and geometric tangent stiffness matrices,
respectively. The contribution $D_i$ resulting from deformation dependent loads will be discussed in more detail in Section 5.2. Combining Eqs. (2.15) and (2.18) yields

$$ (K_c + K_p) \Delta u = -r $$

(2.21)

from which the classical iterative Newton-Raphson procedure can be derived. In each Newton step, the linearized system Eq. (2.21) is solved for the Newton correction $\Delta u$, which is used to update the total displacements $u$, until the norm of the residual vector $r$ has converged below a very small tolerance close to zero.

3. The concept of hierarchical B-spline approximation for interface problems

The present section introduces the fundamental principles of the hierarchical B-spline FEM in 1 dimension, i.e. a regular grid of knot span elements, hierarchical refinement towards a discontinuity, and exponential rates of convergence with an $hp$-refinement strategy.

3.1. B-spline finite elements on regular grids in 1D

A B-spline basis function $N_{i,p}$ of polynomial degree $p$ is defined by $p+2$ knots $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_{p+2}$ in the parameter space $\xi$. The resulting $p+1$ knot spans contain piecewise polynomials of degree $p$, which join smoothly up to a continuous differentiability of $C^{p-1}$. A number of $n$ basis functions constitute a patch, defined by knot vector $\Xi$

$$ \Xi = [\xi_1, \xi_2, \ldots, \xi_{n+p+1}] , \quad \xi_1 \leq \xi_2 \leq \ldots \leq \xi_{n+p+1} \quad (3.1) $$

The knots of each individual basis function $N_{i,p}$ with patch index $i$ can be identified as the consecutive entries $[i, i+1, \ldots, i+p+1]$ in $\Xi$. A knot can be repeated at the same coordinate up to $k=p$ times, which reduces the differentiability of the B-spline patch at the knot position to $C^{p-k}$. B-spline basis functions $N_{i,p}$ of arbitrary polynomial degree $p$ can be generated recursively with the Cox-deBoor formula, starting from piecewise constants $N_{i,0}$.

$$ N_{i,0} (\xi) = \begin{cases} 1, & \text{if } \xi_i \leq \xi \leq \xi_{i+1} \\ 0, & \text{otherwise} \end{cases} \quad (3.2) $$

$$ N_{i,p} (\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1} (\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1} (\xi) \quad (3.3) $$

Uniform B-splines of different polynomial degree $p$ are plotted in Fig. 2. Detailed information on B-splines and their generalization to NURBS can be found for example in Piegl & Tiller (1997), Rogers (2001), Farin (2002) or Cottrell et al. (2009).

A uniform B-spline patch with open knot vectors constitutes a suitable finite element basis, since it guarantees optimal approximation for smooth problems due to maximum continuity within the patch, but also allows for the imposition of boundary conditions by standard FE techniques (Hughes et al., 2005). Its basis functions away from the boundary of the patch consist of uniform B-splines constructed from equidistant knots, which can be interpreted as translated copies of each other (Höllig, 2003). At the boundaries, knots are repeated $p+1$ times in order to make the basis interpolatory. The corresponding entries $\Xi [i]$ of the open knot vector can be summarized as follows

$$ \Xi [i] = \begin{cases} 0, & \text{if } 1 \leq i \leq p + 1 \\ i - (p + 1), & \text{if } p + 2 \leq i \leq n \\ n - p, & \text{if } n + 1 \leq i \leq n + p + 1 \end{cases} \quad (3.4) $$

![Figure 2: B-spline basis functions of increasing polynomial degree $p$, starting from a piecewise constant at $p=0$](image-url)
where \( n \) denotes again the number of basis functions in the patch. An example of an open B-spline patch for polynomial degree \( p=3 \) is shown in Fig. 3a.

B-spline functions resulting from Eq. (3.4) can be inserted into Eqs. (2.12) to (2.14) for the discretization of kinematical quantities in the framework of geometrically nonlinear elasticity as discussed in Section 2. Knot spans can be identified as finite elements in the classical sense, and each knot span element is equipped with full Gaussian integration, i.e. \( p+1 \) integration points (Szabó & Babuška, 1991; Hughes, 2000). The discretization of a one-dimensional physical domain is accomplished with a regular grid of knot span elements of width \( h \), which can be generated from a simple linear transformation of the parameter space \( \xi \) to the physical coordinates \( X \) in the reference configuration

\[
X = X_0 + h \xi
\]

where \( X_0 \) denotes the origin of the reference configuration in the parameter space. The geometry description of the spatial configuration and the Jacobian can thus be computed straightforward as

\[
x = X + \sum_{a=1}^{n} N_{a,p} \, u_a
\]

\[
\frac{\partial x}{\partial \xi} = h + \sum_{a=1}^{n} \frac{\partial N_{a,p}}{\partial \xi} \, u_a
\]

### 3.2. Local refinement with hierarchical B-splines

An essential requirement of finite element schemes is the possibility of adaptively refining the grid, which helps to accurately resolve local characteristics of the solution with a restricted amount of additional degrees of freedom. In the framework of B-spline finite elements, local refinement can be based either on the recent T-spline technology (Sederberg et al., 2004; Bazilevs et al., 2010; Dörfel et al., 2010) or on hierarchical methods (Forsey & Bartels, 1988; Kraft, 1997; Krysl et al., 2003). For interface problems, local refinement is required around the interface to better resolve the discontinuity, for which a hierarchical approach is derived in the following.
3.2.1. Refinement based on B-spline subdivision

A general refinement scheme can be motivated in a natural way by using the subdivision property of B-splines

\[ N_{i,p}^{h} = 2^{-p} \sum_{j=0}^{p+1} \binom{p+1}{j} h_{2i+j,p}^{h/2} \]  

(3.8)

The subdivision formula Eq. (3.8) states that a B-spline of degree \( p \) and grid width \( h \) can be expressed as a linear combination of several B-splines of the same degree, but half the grid width \( h/2 \) (Zorin et al., 2000; Kobbelt, 2002; Höllig, 2003). This is illustrated in Fig. 4 for uniform quadratic and cubic B-splines. Thus, local grid refinement can be achieved as follows:

1. In each refinement step \( k \), select those B-spline basis functions from the patch that have support within the region, where local refinement is required.
2. Replace them with the same B-spline basis functions of half the grid width in the sense of Eq. (3.8). Depending on the refinement step \( k \) and the initial grid width \( h \), the refined grid width \( h_k \) is

\[ h_k = 2^{-k} h, \quad 1 \leq k \leq m \]  

(3.9)

3. Repeat this process until the maximum refinement step \( k=m \) has been reached.

Subdivision based local refinement is illustrated in Fig. 5a, where refined basis functions are arranged in several levels according to refinement step \( k \). The resulting grid consists of a nested sequence of bisected elements, which keep the regular grid property in the sense of Eq. (3.9). In the sense of Fig. 5, levels \( k<i \) and \( k>i \) will be referred to in the following as higher and lower levels, respectively, with respect to the current level \( k=i \). It is important to note that due to Eq. (3.8), the subspace \( B_{k-1} \), representing all basis functions of grid width \( h_{k-1} \) to be refined, is a subset of the subspace \( B_k \), representing corresponding refined basis functions of grid width \( h_k \)

\[ B_{k-1} \subset B_k \]  

(3.10)

Therefore, in each refinement step \( k \), all basis functions of subspace \( B_{k-1} \) have to be removed from the finite element basis in order to preserve its linear independence.

Applying the subdivision property Eq. (3.8) to uniform basis functions of a B-spline curve or surface, it turns out that the coefficients of the refined basis (called control points in this context) can be easily found by forming successive averages of the original control points. If subdivision is repeated and control points are associated with the midpoints of the B-spline supports, the control polygon converges to the actual B-spline curve. In computer-aided geometric design and computer graphics, this led to the development of highly efficient subdivision algorithms for the approximation of smooth surfaces by control meshes (see for example Zorin et al. (2000), Kobbelt (2002) or Warren & Weimer (2002)).
3.2.2. Hierarchical refinement for interface problems

A disadvantage of the subdivision based refinement is the abundance of refined basis functions located further away from the interface. To restrict the amount of additional degrees of freedom, refinements need to be concentrated in the close vicinity of the discontinuity, which can be achieved by modifying the previous scheme as follows:

2. In refinement step \( k \), consider only 3 refined B-splines of grid width \( h_k \), which consists of the B-spline centered to the knot span element of grid width \( h_{k+1} \), where the interface is located in, and its left and right neighbor. Do not replace B-splines of higher levels, but add the 3 refined B-splines to the finite element basis.

The modified refinement scheme, adding groups of 3 contracted B-splines around the interface, is illustrated in Fig. 5b. The choice of the central B-spline and its two neighbors guarantees that the refined basis functions symmetrically embed the knot span element of the higher level, which contains the interface. Since the construction of a B-spline of level \( k \) centered to a knot span of level \( k+1 \) requires that it is defined over an even number of contracted knot spans \( p+1 \) (see Figs. 4a and 4b), the hierarchical B-spline FEM uses B-splines of odd polynomial degrees \( p=1,3,5,... \) in the following. The present refinement scheme is closely related to the \( hp-d \) method, which implements a very similar refinement process for the Legendre basis of the \( p \)-version of the FEM (Rank, 1992; Duster et al., 2007). It will be shown later that the choice of 3 contracted B-splines per level is optimal in comparison to less or more B-splines.

In contrast to Eq. (3.10), subspace \( B_{k-1} \) is now not a subset of subspace \( B_k \) anymore, but the complete refined finite element space \( B \) arises from the union of all subspaces

\[
B : B_0 \cup B_1 \cup ... \cup B_m
\]

(3.11)

Note that Eq. (3.11) restricts the amount of B-splines for each interface and refinement level to a maximum of \( p+1 \) basis functions. In case that a group of refinement functions contains \( p+2 \) or more functions, the finite element basis is not linearly independent, since one B-spline of higher level can be represented as the sum of its refinements by Eq. (3.8). On the other hand, B-splines of higher levels cannot be taken out of the finite element basis, if they are not representable by some combination of lower level B-splines in the sense of Eq. (3.8), since this leads to the loss of completeness of the finite element basis with respect to a first order polynomial.
It is important to note that there are special cases, where the refinement strategy described here still leads to a finite element basis with linear dependencies. For example, consider a knot span discretization of polynomial degree \( p = 3 \), where two interfaces are located in two neighboring knot span elements. With the refinement procedure according to Fig. 5b, a number of \( p+2 = 5 \) consecutive B-splines of the next refinement level \( k+1 \) are generated, which can represent one of the B-spline basis functions of level \( k \) according to Eq. (3.8). Increasing the number of neighboring knot spans with interfaces, this situation can be easily constructed for any polynomial degree \( p \). In this case, one has to go back to the refinement strategy described in the previous sub-section (see Fig. 5a), which identifies all linear dependent B-spline basis functions of level \( k \) and erases them from the basis.

### 3.3. Global h- and p-refinement strategies

Global refinement of the B-spline patch can be achieved by increasing the number of B-splines \( n \) in Eq. (3.4) (h-refinement). For the example patch shown in Fig. 3a, h-refinement is illustrated in Fig. 3b, which doubles the number of knot spans in the patch. The alternative \( p \)-refinement strategy is based on the global increase of the polynomial degree \( p \) of the B-spline functions throughout the patch. However, pure \( p \)-refinement implies a decrease in knot spans, since the number of unmarked knots depends on \( p \) as prescribed by the second line of Eq. (3.4), so that the corresponding physical knot span width \( h \) of Eq. (3.5) must be enlarged to be able to discretize the same domain. As a side effect, this also leads to a gradual delocalization of local hierarchical B-spline refinement, since the physical knot span width \( h_k \) of level \( k \) in Eq. (3.9) is also enlarged in every \( p \)-refinement step. Therefore, global \( p \)-refinement as applied in this paper additionally requires that the number of uniform B-splines in the patch is held constant, while increasing the polynomial degree \( p \). This additional requirement leads to an increase in knot spans, since one additional unemitted knot is added in each \( p \)-refinement step as illustrated in Fig. 3c, and thus prevents the delocalization of hierarchical refinement levels. Note that this form of \( p \)-refinement is similar to \( k \)-refinement, discussed for example in Hughes et al. (2005), Cottrell et al. (2007), Hughes et al. (2008) or Beirão da Veiga et al. (2011).

### 3.4. Adaptive integration of discontinuous knot span elements

The accuracy of Gauss quadrature and related integration schemes, which assume smoothness of the integrands, is considerably influenced by discontinuities within elements. Therefore, most standard discontinuous finite element schemes use sub-elements fitted to the interface, on which integration can then be accomplished by standard methods (see Fries & Belytschko (2010) and the references therein). Since this requires an explicit interface parameterization to be avoided here, the hierarchical B-spline FEM adopts the so-called Gauss point method (GPM), which is used for example in the unaligned method (Zohdi & Wriggers, 2005) or the Finite Cell Method (Parvizian et al., 2007; Düster et al., 2008). GPM relies on the aggregation of Gauss points around the interface to accurately capture its influence on the solution. This can be achieved either very simply, for example by an excessive increase of the order of the Gauss quadrature rule in the elements cut by the geometric boundary (Zohdi & Wriggers, 2005), or in a more sophisticated way by introducing a sub-element based partitioning scheme, which approximates the geometric boundaries by unfitted sub-elements, each of which applies the standard quadrature rule (Schillinger et al., 2010). In the present case of the hierarchical B-spline FEM, the knot span of the finest refinement level \( k \) is treated as a classical finite element with full Gaussian integration. Therefore, Gauss points are automatically aggregated adaptively in the close vicinity of the discontinuity with increasing number of hierarchical refinement levels. It should be noted here that recent results of Hughes et al. (2010) show that the number of integration points required for high accuracy is lower than \( p+1 \) Gauss points in each knot span element. Incorporating an integration scheme with less integration points can considerably increase the computational efficiency of the current scheme, but is not yet incorporated at the present state of development.

### 3.5. Computational performance for the bi-material rod example

The numerical performance of the hierarchical B-spline FEM in one dimension is examined with the bi-material rod example given in Fig. 6. Its geometrically nonlinear formulation according to Section 2.2 simplifies considerably to

\[
\Psi = \frac{E_k}{2} (\ln \lambda)^2 
\]

\[
\sigma = \frac{E_k}{J} \ln \lambda 
\]
resulting error plot Fig. 8a illustrates that independent of the polynomial degree $p$ of the basis, a refinement group size of 3 offers the best trade-off between decrease in error and increase in degrees of freedom.
The second numerical test examines the convergence behavior, if a fixed number of local hierarchical refinement levels is combined with global $h$-refinement of the cubic B-spline patch. The corresponding error plot Fig. 8b illustrates that the total error can be considerably decreased by local hierarchical refinement, but the rate of convergence, being $q=3$ for the cubic basis in the optimal case, is still heavily slowed down by the discontinuity ($q\approx 0.6$). To get an idea of the integration error, the hierarchical B-spline FEM is combined with an overkill Gauss point method (GPM), using 50 Gauss points in the knot span element, where the discontinuity is located. Fig. 8b shows that the integration error for the case of global $h$-refinement plays a significant role and is only slightly improved with increasing number of local hierarchical refinement levels.

The third numerical test examines the convergence behavior, if a fixed number of local hierarchical refinement levels is combined with global $p$-refinement, where the polynomial degree is increased from $p=3$ to 15 in steps of 2. The corresponding error plot Fig. 8c illustrates that both the total error and the convergence rate can be considerably improved by local hierarchical refinement. Super-linear rates are possible (12 local refinement levels achieve $q=8.25$), but the optimal exponential rate of convergence of $p$-refinement cannot be attained. Fig. 8c also shows the results for global $p$-refinement in combination with local subdivision based refinement as shown in Fig. 5a. While the convergence behavior is basically the same, the convergence rates are considerably slower due to the abundance of
Figure 8: Error and convergence in strain energy for the bi-material rod with different combinations of global and local refinements

The fourth numerical test examines the convergence behavior, if the number of local hierarchical refinement levels is simultaneously increased along with global $h$-refinement of the cubic patch or global $p$-refinement. This means that one additional group of 3 contracted B-spline basis functions is added, when the number of knot span elements is doubled or the polynomial degree $p$ is increased, respectively. The corresponding error plot Fig. 8d illustrates that in the case of $h$-refinement, the rate of convergence of the cubic basis is now considerably increased from $q\approx 0.6$ in Fig. 8c to $q=1.08$. In the case of $p$-refinement, the exponential rate of convergence can now be achieved despite the presence of the discontinuity, where the convergence rate continuously increases from around $q\approx 1$ to $q=10.3$. Taking into account the relation between total error, rate of convergence and necessary amount of degrees of freedom in Figs. 8b to 8c, the $hp$-approach, combining simultaneous increase of local hierarchical refinement and global polynomial degree $p$, proves to be the best strategy for the present 1D example.
4. A geometrically nonlinear FCM version, based on hierarchical B-splines

In the following, the hierarchical B-spline approach described in the previous section is combined with the Finite Cell concept. The resulting hierarchical B-spline version of the FCM is shown to be a suitable way to tackle geometrically nonlinear fictitious domain problems.

4.1. The fictitious domain concept and the Finite Cell Method

The recently developed Finite Cell Method (FCM) combines the \( \mathit{p} \)-version of the finite elements with the fictitious domain concept. It allows for easy and fast meshing of complex domains, while preserving the exponential rate of convergence for smooth problems of linear elasticity (Parvizian et al., 2007; D"uster et al., 2008). Its main component is the assumption of a fictitious domain \( \Omega_{fict} \), extending the physical domain \( \Omega_{phys} \) beyond its potentially complex boundaries into a larger embedding domain, which can be meshed easily by a regular grid of \( \mathit{p} \)-elements (see Fig. 9), denoted as finite cells in this context. To preserve consistency with the original problem, Young’s modulus \( E_{fict} \) in the fictitious domain is penalized by parameter \( \alpha \) to be much smaller than \( E_{phys} \) in the physical domain.

\[
E_{fict} = \alpha \cdot E_{phys} \geq 10^{-6} E_{phys} \tag{4.1}
\]

Due to the small \( \alpha \), shape functions with little support in \( \Omega_{phys} \), lead to stiffness matrix entries, which are considerably smaller than entries resulting from shape functions located completely inside \( \Omega_{phys} \). In conjunction with \( \mathit{p} \)-refinement, this leads to a considerable increase of the corresponding condition number, so that a solution with iterative solvers requires a prohibitive number of iterations. Therefore, direct solvers will be applied in the following. If \( \alpha \) is smaller or even zero, the condition number degrades to such an extent that a solution is impossible even with direct solvers. Therefore, \( \alpha \) needs to be bounded to few orders of magnitude to prevent extreme ill-conditioning of the stiffness matrix. Since the smallest support of \( \Omega_{phys} \) in a finite cell, controlling the conditioning of the complete stiffness matrix, is different in each discretization, the smallest \( \alpha \) possible depends on the geometry of the problem and the discretization used. It usually ranges between \( \alpha = 10^{-3} \) and \( 10^{-6} \).

The FCM solution behavior as shown in Parvizian et al. (2007) and D"uster et al. (2008) can be outlined with the help of the best approximation property to the total strain energy, which states that the solution of a Galerkin finite element scheme represents a least-squares best fit to the exact solution in terms of the strain energy integral over the complete domain \( \Omega \) (Hughes, 2000; Zienkiewicz & Taylor, 2000). Due to the penalization with parameter \( \alpha \), deviations from the exact solution in \( \Omega_{fict} \) have a considerably smaller impact on the strain energy than deviations in \( \Omega_{phys} \). Therefore, a minimization of the strain energy error by the high-order basis of the FCM scheme results in an accurate approximation in \( \Omega_{phys} \), where potential deviations lead to considerable error contributions, and a largely inaccurate approximation in \( \Omega_{fict} \), where potential deviations lead to negligible error contributions due to penalization. When implemented in a geometrically nonlinear formulation as described in Section 2, FCM yields solution fields with large oscillations throughout the discontinuous cells, which can be traced back to the nonlinear strain measure. Whereas engineering strains of linear elasticity are bound by definition to very small values \( |\varepsilon_{lin}| \ll 1 \), nonlinear strains are able to grow without bounds in order to yield physically meaningful measures for very large deformation states. However, in the case of large deformations in the fictitious domain, nonlinear strains thus act as a counterbalance to \( \alpha \) and increasingly outweigh the penalization of \( \Omega_{fict} \). As a consequence, the contribution of \( \Omega_{fict} \) to the total strain energy grows, so that FCM tries to accurately fit the solution in both the fictitious and physical domains due

\[\text{Figure 9: The fictitious domain approach allows for easy meshing of complex geometries}\]
Figure 10: Stress plots for the 1D fictitious domain example. The \( p \)-version uses 3 equidistant elements of \( p=10 \), whereas the hierarchical B-spline version \( p=5 \) and 4 local refinement levels.

Figure 11: Convergence for the standard \( p \)-version and hierarchical B-spline version of the FCM. Note that the total strain energy of the FCM schemes takes into account only \( \Omega_{\text{phys}} \).

to the best approximation property. However, this leads to large oscillations in high-order finite elements with inter-element discontinuities and the decay of the strain energy convergence to a low algebraic rate, which is a well-known issue since the early days of \( p \)-FEM (Szabó & Babuška, 1991).

In order to illustrate the oscillatory behavior in case of large deformation states in \( \Omega_{\text{fict}} \), the rod of Fig. 6 is modified as follows. Decreasing Young’s modulus \( E_2 \) to \( E_{\text{fict}}=10^{-4} \), the strong and weak parts \( R_1 \) and \( R_2 \) can be interpreted as physical and fictitious domains, respectively, with a geometric boundary at \( X=1.12 \). In addition to the load \( f_{\text{sin}} \), which is only applied in \( \Omega_{\text{phys}} \), a large deformation in the fictitious domain is induced by displacement \( u_{\text{fict}} \) at the right boundary. The physical reference system thus consists of rod \( R_1 \), loaded by \( f_{\text{sin}} \) and a single compressive load

\[
F = -\ln \left( \frac{u_{\text{fict}}}{L_2} \right) \frac{u_{\text{fict}}}{L_2}
\]

(4.2)
at the geometric boundary, which is equivalent to the deformation in \( \Omega_{\text{fict}} \). The reference strain energy is again determined by an overkill discretization of 1000 cubic elements as \( U_{\text{ref}} = 4.74371354 \cdot 10^{-4} \), taking into account only the physical system. A FCM discretization of 3 equidistant \( p \)-elements over the complete domain \( \Omega \) yields oscillatory stresses as shown in Fig. 10, which occur in the center element, where the geometric boundary is located. Due to their influence on the strain energy, increasing the polynomial degree \( p \) achieves only a slow algebraic rate of convergence around \( q \approx 0.7 \) (see Fig. 11).

4.2. The hierarchical B-spline version of the Finite Cell Method

Interpreting the embedding domain as a two-phase material with significant difference in Young’s modulus \( E \), the derived hierarchical B-spline FEM can be applied straightforward in a FCM sense, since it exhibits all relevant properties, i.e. high-order basis functions on a regular grid and adaptive integration (Schillinger et al., 2010). Additionally, it provides the possibility of hierarchical refinement around material interfaces, which correspond to geometric boundaries in the fictitious domain context. The resulting hierarchical B-spline version of the FCM can thus effectively limit the oscillatory behavior in the solution fields to the close vicinity of geometric boundaries and preserves the exponential rate of convergence in terms of the strain energy measure. At the same time, it keeps the key advantage of the FCM in terms of the simple parameterization of complex domains, since refinements preserve the regular grid structure of FCM and can be easily generated from explicit geometric representations with the algorithm described in Section 3.2. This is demonstrated for the 1D fictitious domain example, for which the discretization of Fig. 3 is used as an unrefined starting point. Fig. 10 shows the effective reduction of oscillations in stresses, if only 4 levels of hierarchical refinements are added. With the \( hp \)-strategy, which simultaneously increases the number of hierarchical levels with the polynomial degree \( p \) of the B-spline basis, high rates of convergence can be restored with a maximum of \( q = 3.38 \) (see Fig. 11). However, the convergence behavior does not exhibit exponential rates of convergence as
in the original version of the FCM for linear problems (Parvizian et al., 2007; Düster et al., 2008). The hierarchical B-spline version of the FCM thus provides a possible way to extend the FCM concept to geometrically nonlinear problems, while still preserving high rates of convergence. Further examples will be presented in Section 7.

5. The hierarchical B-spline FEM for interface problems in 2 and 3 dimensions

The extension of the one-dimensional hierarchical B-spline FEM to 2 and 3 dimensions is straightforward, since the simplicity of hierarchical B-spline refinement directly carries over to higher dimensions. Due to the regular grid structure, special strategies are required for general boundary conditions along arbitrary curves. In the framework of geometric nonlinearity, the implementation of deformation dependent loads is also discussed.

5.1. Tensor product B-splines and their hierarchical refinement

A multivariate B-spline basis can be easily constructed by products of corresponding univariate B-spline basis functions (Höllig, 2003; Hughes et al., 2005; Cottrell et al., 2009). The resulting tensor product structure of multivariate B-splines allows for a straightforward generalization of all univariate properties and algorithms. Analogous to the univariate case, the starting point is the following open knot vector structure

\[ \Xi_\theta [i] = \begin{cases} 0, & \text{if } 1 \leq i \leq p + 1 \\ i - (p + 1), & \text{if } p + 2 \leq i \leq n_\theta \\ n_\theta - p, & \text{if } n_\theta + 1 \leq i \leq n_\theta + p + 1 \end{cases} \] (5.1)

Depending on the dimensionality, \( \theta \) represents the independent directions of the two- or three-dimensional parameter space \( [\xi, \eta] \) and \( [\xi, \eta, \zeta] \), respectively, and \( n_\theta \) determines the number of univariate B-spline basis functions in each direction. The two- or three-dimensional parameter space can thus be discretized by a structured grid of knot span elements, whose nodal positions can be obtained by permuting all entries \( i, j, k \) of Eq. (5.1) in \( [\xi, \eta, \zeta] \) directions, respectively. Corresponding multivariate B-spline basis functions are then obtained in 2 and 3 dimensions, respectively, by taking the tensor product of its univariate components of each direction \( \theta \)

\[ N_{i,j,k} (\xi, \eta, \zeta) = N_{i,j} (\xi) \cdot N_{j,k} (\eta) \cdot N_{k} (\zeta) \] (5.2)

\[ N_{i,j,k} (\xi, \eta, \zeta) = N_{i} (\xi) \cdot N_{j} (\eta) \cdot N_{k} (\zeta) \] (5.3)

An example of two dimensional knot span elements and a corresponding bi-cubic uniform B-spline are shown in Fig. 12. In 2 and 3 dimensions, each knot span element can be identified as a quadrilateral or hexahedral finite element, respectively, with full Gaussian integration (Höllig, 2003; Hughes et al., 2005; Cottrell et al., 2009).

In analogy to the one-dimensional case of Eq. (3.5), the regular grid discretization allows for a straightforward transformation of the three-dimensional parameter space to the physical reference configuration \( X \) by

\[ X = \begin{bmatrix} X_0 + h_\xi, \xi \\ Y_0 + h_\eta, \eta \\ Z_0 + h_\zeta, \zeta \end{bmatrix} \] (5.4)

\( X_0, Y_0, Z_0 \) denote the origin of the reference configuration in the parameter space, and \( h_\xi, h_\eta, h_\zeta \) are the grid widths in Cartesian coordinate directions. The corresponding two-dimensional transformation can be readily obtained by omitting the third line of Eq. (5.4). The current position \( x \) in the spatial configuration is obtained by adding the current displacements \( u \) in the discretized form of Eq. (2.12)

\[ x = X + \sum_{\alpha=1}^{n} N_{\alpha} u_{\alpha} \] (5.5)

from which the Jacobian can be obtained in the standard way (Bonet & Wood, 2008; Wriggers, 2008).

The hierarchical refinement strategy for interface problems introduced in Section 3.2 can be generalized straightforward for higher dimensions (Kraft, 1997; Krysl et al., 2003). Analogous to the one-dimensional definition, interfaces in 2 or 3 dimensions, which occur in the form of curves or surfaces, respectively, are refined as follows:
1. Traverse all knot span elements of the currently finest level. Check material properties at each integration point.

2. If integration points of the same knot span element are associated with different material properties (hence, a material interface must be present), find in each parameter direction θ the 3 univariate B-splines of the next refinement level, consisting of the contracted B-spline centered to the knot span element as well as its two direct neighbors.

3. Use the univariate contracted B-splines to construct the corresponding tensor product B-splines (3×3 in 2 dimensions, 3×3×3 in 3 dimensions) of half the grid width

\[ h_{x,k} \times h_{y,k} = 2^{-k}h_x \times 2^{-k}h_y \]  
\[ h_{x,k} \times h_{y,k} \times h_{z,k} = 2^{-k}h_x \times 2^{-k}h_y \times 2^{-k}h_z \]

where \( h_x, h_y, h_z \) are the initial grid widths, and add them as additional basis functions to the finite element basis.

4. Provide all new knot span elements of the currently finest level with \((p+1)\times(p+1)\) or \((p+1)\times(p+1)\times(p+1)\) integration points and corresponding material parameters.

5. Repeat this process, until a sufficient refinement level is reached.

In this form, hierarchical B-spline refinement is easy to implement, fast to generate, and especially well suited, if an implicit or voxel based representation of the geometry (Bungartz et al., 2004) is available (see Section 6). It is illustrated for the two-dimensional case of a rectangular plate with a circular inclusion in Fig. 13.

5.2. Linear independence in 2 and 3 dimensions

In 2 and 3 dimensions, the subdivision property Eq. (3.8) applies in each parametric direction due to the tensor product structure of the multivariate basis. Therefore, a minimum of \( p+2 \) consecutive refined B-splines in each parametric direction are necessary to represent at least one B-spline of level \( k \) by a combination of refined B-splines of level \( k+1 \), which leads to the loss of linear independence. However, since the refinement procedure applied to each knot span element cut by an interface adds only 3 refined B-splines in each parametric direction, which normally does not lead to enough consecutive refined B-splines of level \( k+1 \), linear independence of the refined basis will automatically be satisfied in almost all cases. Analogous to the 1D example (see Section 3.2.2), there exist special situations, where a loss of linear independence can still occur. For instance in the case of a cubic knot span discretization, a refinement of 2 neighboring knot span elements in each parametric direction is sufficient to provide enough consecutive refined B-splines of level \( k+1 \), which can happen in the presence of very close interfaces, very small inclusions or very sharp turns of interfaces. Therefore, additional care must be taken to properly identify these situations in each hierarchical level of the refined basis, which can be easily achieved by counting the consecutive refined knot spans in each parametric direction. Recovering a linearly independent basis can then be achieved by removing all linear dependent B-splines of level \( k \) from the basis as illustrated in Fig. 5a for the 1D case.
5.3. Boundary conditions and deformation dependent loads

In case of rectangular problems, whose domain conforms to the boundary of the regular B-spline grid, boundary conditions can be implemented in the standard way due to the open knot vector structure of Eq. (5.1) (Hughes et al., 2005; Cottrell et al., 2009). For problems of arbitrary geometry, the domain of interest cannot be fitted to the boundary of the regular grid. Corresponding boundary conditions, defined along arbitrary curves that cut the regular B-spline discretization, can be either incorporated by variational methods (see for example Glowinski & Kuznetsov (2007) or Embar et al. (2010)), by direct manipulation of the cut B-spline shape functions (Höllig et al., 2002; Höllig, 2003; Sanches et al., 2011) or by penalty methods, which are adopted in the scope of the present paper in the form of the fictitious domain concept as introduced in Section 4. To this end, the complete domain Ω covered by the regular B-spline discretization is separated in a physical domain Ωphys and a fictitious domain Ωfict according to Fig. 9, where material parameters are penalized in the sense of Eq. (4.1).

5.3.1. Dirichlet BC and Neumann BC of zero traction

In case of homogeneous Dirichlet boundary conditions, Young’s modulus $E_{\text{fict}}$ is chosen some orders of magnitude stiffer than $E_{\text{phys}}$, so that Dirichlet conditions applied to the boundaries of the regular grid are directly carried on to the boundary of $\Omega_{\text{phys}}$ (Ramière et al., 2007; Düster et al., 2008). For Neumann conditions of zero traction, Young’s modulus $E_{\text{fict}}$ is chosen some orders of magnitude softer than $E_{\text{phys}}$. Thus, stresses cannot be transferred beyond the physical domain, since $\Omega_{\text{fict}}$ lacks the stiffness to support them. Numerical experiments indicate that penalizations of at least three orders of magnitude are necessary to sufficiently strengthen or diminish the influence of $\Omega_{\text{fict}}$. Algorithmically, the fictitious domain approach for the incorporation of boundary conditions along arbitrary curves fits well in the hierarchical B-spline FEM, since the fictitious domain can be interpreted as an additional material phase, whose interfaces can be resolved by hierarchical refinement to accurately capture the corresponding discontinuities.

5.3.2. General Neumann boundary conditions

In contrast to Neumann boundary conditions of zero traction, those of non-zero traction additionally require an integration over the boundary $\Gamma_N$, which therefore needs to be parameterized as a curve or surface. Furthermore, in the framework of geometric nonlinearity, the deformation dependence of the loading needs to be taken into account (Schweizerhof & Ramm, 1984; Bonet & Wood, 2008; Wriggers, 2008).

In the scope of the present paper, the implementation of a body attached normal pressure $\hat{p}$ along a two-dimensional polygon is discussed (Mok et al., 1999). The procedure described in the following can be analogically applied to three-dimensional problems by introducing for example a triangular approximation for 3D boundary surfaces of arbitrary...
geometry. From Eq. (2.11), the virtual work due to \( \hat{\varphi} \) acting on \( \Gamma_N \) follows with unit normal vector \( \mathbf{n} \) and traction vector \( \mathbf{t} = \hat{\varphi} \mathbf{n} \) as

\[
\delta W(\varphi, \delta \varphi) = \int_{\partial \Gamma_N} \hat{\varphi} \mathbf{n} \cdot \delta \varphi \, d\mathbf{a}
\]

(5.8)

The standard algorithmic treatment of Eq. (5.8) is based on a re-parameterization of the unit normal vector \( \mathbf{n} \) in terms of convective coordinates of the boundary curve (see Schweizerhof & Ramm (1984); Simo et al. (1991); Mok et al. (1999) for an in-depth discussion). In the two-dimensional case, this leads to a normal vector \( \hat{\mathbf{n}} \), which can be calculated pointwise from the gradient of the deformed position \( \mathbf{x}' = [x', y']^T \) of the curve \( \mathbf{s} \) with respect to the coordinate \( \varphi \) of its parameter space as follows

\[
\hat{\mathbf{n}}_i = \epsilon_3 \times \frac{\partial \mathbf{x}'_s}{\partial \varphi} = \begin{bmatrix} -y_s' \\ x_s' \end{bmatrix}
\]

(5.9)

In standard finite elements, where boundaries are explicitly parameterized in terms of a local coordinate \( \xi \) along an element edge, Eq. (5.9) can be obtained on the basis of the isoparametric concept (Mok et al., 1999; Wriggers, 2008). The hierarchical B-spline FEM, however, defines a general Neumann boundary independently of the structured knot span discretization. \( \Gamma_N \) can thus be an arbitrary curve, approximated with arbitrary precision by the polygon

\[
\mathbf{s} : \begin{bmatrix} X_s \\ Y_s \end{bmatrix} = \sum_{i=1}^{n_{vert}} N_{i^{lin}} (\varphi) \cdot \begin{bmatrix} X_i \\ Y_i \end{bmatrix}
\]

(5.10)

where \( X_i \) and \( Y_i \) denote the physical coordinates of the \( n_{vert} \) vertices of the polygon in the reference configuration, which lie on \( \Gamma_N \). Nodal basis functions \( N_{i^{lin}}(\varphi) \) defined over parameter space \( \varphi \in [-1, 1] \) linearly interpolate \( \Gamma_N \) between vertices. Assuming that wherever polygon \( \mathbf{s} \) crosses an element edge, a vertex is placed, each polygon segment can be uniquely related to a knot span element, and the parameterization of \( \mathbf{s} \) in terms of local coordinates \( [\xi, \eta] \) follows as

\[
\mathbf{s} : \begin{bmatrix} \xi_s' \\ \eta_s' \end{bmatrix} = \sum_{i=1}^{n_{vert}} N_{i^{lin}} (\varphi) \cdot \begin{bmatrix} \xi_i' \\ \eta_i' \end{bmatrix}
\]

(5.11)

where \( \xi_i' \) and \( \eta_i' \) denote the local coordinates of \( X_i' \) and \( Y_i' \) in the corresponding knot span element. The unit tangent vector \( \mathbf{t} \) in local coordinates for each polygon segment can be computed as

\[
\mathbf{t} = \frac{1}{\sqrt{\Delta \xi^2 + \Delta \eta^2}} \begin{bmatrix} \Delta \xi \\ \Delta \eta \end{bmatrix}
\]

(5.12)

with \( \Delta \xi = \xi_{i+1}' - \xi_i' \) and \( \Delta \eta = \eta_{i+1}' - \eta_i' \). At any point \( [\xi', \eta']^T \) between polygon vertices, the components of the normal vector \( \hat{\mathbf{n}} \), Eq. (5.9) can now be computed by taking the derivative of the geometry in the spatial configuration provided by Eq. (5.5) with respect to parameter \( \varphi \)

\[
\mathbf{x}_{i,\varphi} = \begin{bmatrix} h_i \frac{\Delta \xi}{\Delta \eta} + \sum_{a=1}^{n} N_{a,\varphi} (\xi', \eta') \mathbf{u}_a \\ h_i \frac{\Delta \eta}{\Delta \xi} \end{bmatrix}
\]

(5.13)

\( h_i \) and \( h_i \) denote the Cartesian widths of the regular knot spans in the reference configuration, and \( \frac{\Delta \xi}{\Delta \eta} \) result from the chain rule components \( \frac{\partial \xi}{\partial \varphi} \) and \( \frac{\partial \eta}{\partial \varphi} \), respectively. Eq. (5.13) bears strong similarities to the deformation dependent load implementation of \( p \)-FEM (Yosibash et al., 2007a; Heisserer, 2008). Derivatives of basis function \( N \) with respect to \( \varphi \) can be obtained from the directional derivative along \( \mathbf{t} \) and a chain rule contribution \( \frac{\partial N}{\partial \varphi} \), which is the ratio between the length of the segment in local element coordinates and the length of the parameter space \( \varphi \)

\[
N_{i,\varphi} = \frac{\partial N}{\partial \varphi} \frac{\partial h_i}{\partial \varphi} = \frac{\partial N}{\partial \varphi} \frac{\partial h_i}{\partial \varphi} \frac{\sqrt{\Delta \xi^2 + \Delta \eta^2}}{2} = \frac{\partial N}{\partial \xi} \frac{\Delta \xi}{\Delta \eta} + \frac{\partial N}{\partial \eta} \frac{\Delta \eta}{\Delta \xi}
\]

(5.14)

With known normal vector \( \hat{\mathbf{n}} \), Eq. (5.9), the consistent force vector entries corresponding to each basis function \( N_i \) are obtained by summing up the standard expression as given for example in Mok et al. (1999), Yosibash et al. (2007a) or Wriggers (2008) for each polygon segment
\[ R_a = \sum_{i=1}^{n_{\text{seg}}} \int_{\theta=-1}^{1} \hat{\gamma} N_a \left[ x_{\theta} \right] d\theta \]  

(5.15)

where \( n_{\text{seg}} \) is the total number of polygon segments.

The influence of the deformation dependent load on the stiffness matrix of the linearized system Eq. (2.18) follows from the linearization and discretization of Eq. (5.8) (Bonet & Wood, 2008; Wriggers, 2008)

\[ D\delta \mathbf{W} = \int_{\varphi(\Gamma_N)} \delta \mathbf{v} \cdot \hat{\gamma} p \left( e_3 \times \frac{\partial (\Delta \mathbf{u})}{\partial \vartheta} \right) d\vartheta = \delta \mathbf{v}_a \cdot \mathbf{k}_{ab} \Delta \mathbf{u}_b \]  

(5.16)

From the incremental displacements discretized in the sense of Eq. (2.12), its gradient with respect to \( \vartheta \) can be easily computed analogous to Eq. (5.13) as

\[ \Delta \mathbf{u}_{\vartheta} = \sum_{a=1}^{n} N_{a,\vartheta} \Delta \mathbf{u}_a \]  

(5.17)

where \( N_{a,\vartheta} \) follows from Eq. (5.14). The contribution of each pair of basis functions \( N_a \) and \( N_b \) to the load stiffness matrix follows again by summing up the standard expression as given for example in Mok et al. (1999), Yosibash et al. (2007a) or Wriggers (2008) for each polygon segment

\[ k_{ab} = \sum_{i=1}^{n_{\text{seg}}} \int_{\theta=-1}^{1} \hat{\gamma} N_a N_{b,\vartheta} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} d\vartheta \]  

(5.18)

A numerical example, involving homogeneous Dirichlet and deformation dependent Neumann boundary conditions along circular boundaries, will be shown in Section 7 to illustrate the use of the fictitious domain concept for the implementation of unfitted boundary conditions in geometrically nonlinear structured B-spline discretizations.

6. Geometrical models and hierarchical grid generation

The fundamental advantage of the hierarchical B-spline FEM for interface problems is the very simple and fast grid generation irrespective of the geometric complexity involved. It is based on the disconnection of the regular B-spline discretization from the geometry, which is represented explicitly by the change of material parameters at integration point level. Two explicit geometrical models are introduced here as a basis for fast hierarchical grid generation.

6.1. Analytical inequalities

Analytical inequalities provide a convenient way to implicitly define interface geometries in the form of a hypersurface (Bungartz et al., 2004), e.g.

\[ \frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} \leq 1 \]  

(6.1)

Depending on the parameters \( a, b, c \), Eq. (6.1) represents material inclusions of different geometries, for example a circle of radius \( R \) \((a=b=R; Z=0)\), a sphere of radius \( R \) \((a=b=c=R)\) or an ellipsoid \((a \neq b \neq c)\). If the inequality is satisfied at an integration point, material parameters of the inclusion are valid; otherwise material parameters of the surrounding matrix material are used. Several inequalities with different material parameters can be easily combined to define more complex shapes. Eq. 6.1 will be used in Section 7 to define a series of simple benchmark problems to test the hierarchical B-spline FEM for accuracy and computational efficiency.
6.2. Spatial voxel model

Spatial voxel models are the standard way of explicitly representing very complex geometries in 3 dimensions (Bun-gartz et al., 2004). Voxels are volume elements, aligned in a structured spatial grid in Cartesian directions. Each voxel holds its own material parameters, so that the geometry of material interfaces is explicitly represented by the change of material parameters from one voxel to the next. The geometry error of the voxel model thus depends on the number of voxels resulting from the chosen voxel grid size. Voxel models of complex multiphase materials are usually derived with the help of computed tomography (CT) scans, for example in bio-medical applications (Yosibash et al., 2007b) or material science (Fiedler et al., 2009), but can also be generated from a B-rep geometry of a CAD model (Wenisch & Wenisch, 2004). Spatial voxel models provide an ideal geometrical basis for the generation of refined grids in the framework of the hierarchical B-spline FEM. Assuming a lexicographical ordering of the voxel data, the integration point position \( X, Y, Z \) in physical coordinates can be related to the voxel index \( k_{\text{vox}} \) and the corresponding material parameters by

\[
k_{\text{vox}} = \left\lfloor \frac{(X - X_0)n_x}{L_x} \right\rfloor n_y n_z + \left\lfloor \frac{(Y - Y_0)n_y}{L_y} \right\rfloor n_z + \left\lfloor \frac{(Z - Z_0)n_z}{L_z} \right\rfloor + 1
\]

where \( [X_0, Y_0, Z_0] \), \( [L_x, L_y, L_z] \) and \( [n_x, n_y, n_z] \) denote the origin, the length and the number of voxels in each Cartesian direction, and \( \lfloor \cdot \rfloor \) is the floor function.
The computational efficiency of the voxel approach for hierarchical grid generation is illustrated by a foam composite sample of size $10 \times 10 \times 10$ mm, whose internal geometry information is provided by voxels with a resolution of $2^{10}$ in each Cartesian direction, obtained from a CT scan. The sample consists of two material phases, namely the aluminium foam and the alkyd resin matrix, which are encoded by a material index of 1 or 0, respectively. Fig. 14a shows all voxels of material index 1 associated with the foam in a coarsened resolution of $2^8$ to make single cubes perceptible. The fully automated generation of the corresponding adaptive grid structure, shown in Fig. 14b, with 3 levels of hierarchical refinements and an initial knot span discretization of $8 \times 8 \times 8$ can be accomplished in only 53 seconds. The main costs result from the loading of voxel information encoded by $2^{10}$ bits, the generation of 111,917 adaptive knot span elements and about $10^8$ integration point queries according to Eq. 6.2. Fig. 15a through 15d shows the knot span elements of the different hierarchical levels in the final grid. It can be easily observed that the adaptive aggregation of sub-cells around geometric boundaries increases quickly with level $k$.

7. Numerical examples

The hierarchical B-spline FEM is implemented in the framework of nonlinear elasticity in principal directions as reviewed in Section 2, and its accuracy and computational efficiency are examined for a number of benchmark problems. Using the voxel model of the previous section, it is also applied to simulate a compression test of an aluminium foam sample for the multiphase and the fictitious domain case. Our hierarchical B-spline implementation is largely based on Sandia’s library framework Trilinos (2010) and uses the direct solvers Lapack (2010) and Pardiso (2010), which all should be given credit here. Its results are compared to overkill solutions derived with standard linear quadrilateral and quadratic tetrahedral elements on conforming meshes, provided by the open-source nonlinear finite element code FlagShyp (2008), which is based on the same constitutive and algorithmic formulations (see Section 2). Conforming mesh generation with quadrilaterals and tetrahedrals is accomplished with the software packages Visual DoMesh (2010) and Netgen (2010), respectively.

7.1. Plate with circular inclusion

The most widely used material interface benchmark consists of a 2 dimensional plate in plane stress with a circular inclusion (Sukumar et al., 2001; Mergheim & Steinmann, 2006; Parvizian et al., 2007; Dolbow & Harari, 2009). Material and geometric parameters as well as boundary conditions are given in Fig. 16a. For the computation of the corresponding reference solution, the symmetry in geometry and boundary conditions is made use of to reduce the plate to 1/4 of the original system. The considered FlagShyp discretization with a mesh conforming to the interface consists of 106,674 standard linear quadrilateral elements and 213,681 degrees of freedom. Multiplying the resulting strain energy by 4 yields a reference strain energy of the complete system of $U_{ex} = 4.330559 \cdot 10^{-7}$. Convergence studies with different mesh sizes indicate that the given $U_{ex}$ is correct up to the 6th decimal.

For the hierarchical B-spline computations, the origin of the coordinate system is placed in the center of the circular inclusion, so that its geometry can be described implicitly by $X^2 + Y^2 \leq r^2$ as explained in Section 6.1. For the solution of the geometrically nonlinear system, the displacement load is cut into 5 equally sized increments, each of which requires 3 to 5 Newton-Raphson iterations to converge to a norm of the residual below $10^{-12}$ as described in Section 2.3. The beneficial effect of adaptive hierarchical refinement is illustrated in Fig. 17, which compares Von Mises stresses for an example discretization of $6 \times 6$ knot span elements of polynomial degree $p=5$ with no refinement to a corresponding discretization with $m=4$ adaptive refinement levels around the interface. Whereas the unrefined B-spline solution exhibits large oscillations throughout the domain, the hierarchical B-spline solution accurately localizes the typical stress concentrations along vertical inclusion boundaries, and stresses away from the interface are free of oscillations. Note that already a moderate number $m=4$ of hierarchical refinement levels considerably improves the stress solution. This behavior is further examined in Fig. 18, where Von Mises stresses obtained along cut line A-B (see Fig. 16a) are compared to the reference solution. In analogy to the 1D case discussed in Section 3, adaptive hierarchical B-spline refinement around the interface confines the stress oscillations to the close vicinity of the interface, whereas stresses further away are free of oscillations and fit the reference solution very well.

1 Courtesy of IZFP Fraunhofer Institute for Non-Destructive Testing, Saarbrücken, Germany; http://www.izfp.fraunhofer.de
2 On a Intel(R) Core(TM)2 T5550 @ 1.83 GHz with 3 Gb RAM
Similar to Section 3.5, a convergence study considering the total strain energy examines the effects of $h$-, $p$- and $hp$-refinement strategies. Their effectiveness is again assessed on the basis of the rate of convergence in strain energy according to Eqs. (3.15) and (3.16), where the FlagShyp reference serves as $U_{ex}$. The resulting convergence behavior is shown in Fig. 19. First, global $p$-refinement without adaptive refinement is examined by increasing the polynomial degree from $p=5$ to $15$ on an unrefined regular B-spline discretization of $(p+1) \times (p+1)$ knot span elements. Due to the excessive oscillations in stresses, provoked by the discontinuities at the material interface, the convergence in strain energy only achieves a low algebraic rate of $q \approx 0.33$ at a high error level. Second, $p$-refinement on the same grid is supplemented by a constant number $m=3$ of hierarchical refinement levels around the interface. While the rate of convergence only slightly improves to $q \approx 0.41$, the overall error level is considerably reduced. Third, the effect of local $h$-refinement with hierarchical B-splines is examined, which starts from a cubic B-spline discretization of $6 \times 6$ knot span elements. The polynomial degree is held constant at $p=3$ and one level of hierarchical B-splines is successively added around the interface up to a maximum number of levels $m=8$. Local $h$-refinement with adaptive B-splines achieves a rate of convergence of $q \approx 0.34$.

Finally, $hp$-refinement in the sense of Section 3.5, combining the elevation of the polynomial degree and the addition of hierarchical refinement levels, is tested. The unrefined B-spline discretization consists again of $(p+1) \times (p+1)$ knot span elements and $m=p-1$ hierarchical refinement levels are added around the interface, which results in a maximum rate of convergence of $q \approx 0.82$. The convergence study illustrates that unrefined B-spline discretizations lead to low rates of convergence in strain energy at high error levels. This can be considerably improved by adaptive hierarchical refinement around interfaces, where the best results could be achieved with the $hp$-strategy, which
simultaneously increases the polynomial degree and the number of hierarchical refinement levels.

7.2. Plate with circular hole

Penalizing Young’s modulus $E_2$ with $\alpha = 10^{-4}$ in the sense of Eq. (4.1), the benchmark problem of Fig. 16a turns from an interface problem into a fictitious domain problem. Whereas the plate can be identified as the physical domain $\Omega_{phys}$, the circular domain in the center turns into the fictitious domain $\Omega_{fict}$. The corresponding reference problem is thus a plate with a circular hole, taking into account only the physical domain of the plate. For the computation of the reference solution, the FlagShyp discretization of a quarter of the plate consists of 116,782 standard linear quadrilateral elements with 234,341 degrees of freedom, which leads to a reference strain energy of the complete system of $U_{ex} = 2.305691 \cdot 10^{-3}$.

For the solution of the geometrically nonlinear fictitious domain problem, the $p$-version and the hierarchical B-spline version of the Finite Cell Method as introduced in Section 4 are used. Both methods discretize the physical domain $\Omega_{phys}$ as well as the fictitious domain $\Omega_{fict}$, where the influence of $\Omega_{fict}$ is reduced by penalization parameter $\alpha$. For the present problem, the lower bound of $\alpha$ is $10^{-4}$ to prevent extreme ill-conditioning of the stiffness matrix. The geometric boundary of the circle is again represented implicitly by $X^2 + Y^2 \leq r^2$. As proposed in Parvizian et al. (2007) for the same geometry, the $p$-version of the FCM uses a discretization of $2 \times 2$ finite cells (see Fig. 20a), whose polynomial degree is increased in steps of 2 from $p=2$ to 20. The hierarchical B-spline version of the FCM applies the $hp$-refinement strategy, simultaneously increasing the polynomial degree $p$ and the number $m$ of adaptive hierarchical
refinement levels. In analogy to the interface case of the previous sub-section, the unrefined grid consists of \((p+1) \times (p+1)\) knot span elements as shown in Fig. 17a, which are then refined by several levels of \(m=p-I\) hierarchical B-splines as shown in Fig. 17b.

A first impression of the solution behavior can be obtained from the Von Mises stress plots shown in Figs. 20b to 20d. Whereas the unrefined B-spline discretization yields again very poor results, suffering from large oscillations around the geometric boundary, both \(p\)- and hierarchical B-spline versions of the FCM are able to accurately localize the typical stress concentration phenomenon at the vertical edges of the circular hole and are free of large-scale oscillations in the physical domain of interest. Fig. 21 gives a more detailed view of the Von Mises stresses in the physical domain along cut line A-B (see Fig. 16a) for the \(p\)-version and the hierarchical B-spline version of the FCM. Whereas high-order shape functions of the \(p\)-elements show slight oscillations, leading to a deviation from the reference solution especially in the vicinity of the geometric boundary, the adaptive refinement of the hierarchical B-spline version around geometric boundaries guarantees an accurate stress solution free of oscillations, in particular in the close vicinity of the geometric boundary.

Fig. 22 compares the convergence behavior in strain energy of global \(p\)-refinement with the \(p\)-version of the FCM and \(hp\)-refinement of the hierarchical B-spline version of the FCM for the geometrically nonlinear plate with a circular hole. Due to the influence of the nonlinear strain measure discussed in Section 4, the \(p\)-version cannot reproduce its exponential rate of convergence, which could be obtained with \(2 \times 2\) finite cells for the linear elastic plate with a circular hole as shown in Parvizian et al. (2007). Instead, it shows an algebraic rate of \(q \approx 0.4\). Due to its adaptive refinement around geometric boundaries, the hierarchical B-spline version prevents oscillations in stresses, which results in an improved rate of convergence in strain energy of up to \(q=1.12\). The present example thus illustrates that the hierarchical B-spline FEM is a possible way to effectively transfer the FCM idea to geometrically nonlinear problems, combining its main advantage of easy mesh generation with high convergence rates. However, exponential rates of convergence could not be achieved with the present approach.

7.3. Ring under internal pressure

In Section 5.2, possible methods are discussed to deal with Dirichlet boundary conditions and deformation dependent loads, which do not conform to the boundaries of the B-spline discretization. The 2D example of a circular ring in plane strain under internal pressure (see Fig. 16c) illustrates the capabilities of these formulations in the framework of the hierarchical B-spline FEM. Besides the physical domain \(\Omega_1\), the regular B-spline discretization also covers the fictitious domains \(\Omega_2\) and \(\Omega_3\) due to the curved shapes of the Dirichlet boundary \(\Gamma_D\) and the Neumann boundary \(\Gamma_N\), which do not allow a conforming discretization with a regular B-spline grid. The geometry of the domains enclosed by \(\Gamma_D\) and \(\Gamma_N\) is described implicitly by the analytical inequality \(X^2 + Y^2 \leq r^2\) according to Section 6.1. Homogeneous Dirichlet boundary conditions along \(\Gamma_D\) are realized by fixed displacements at the outer boundaries of the B-spline discretization, which are carried on to \(\Gamma_D\) due to the increased stiffness \(E_3\) in sub-domain \(\Omega_3\), whereas the
Figure 20: Comparison of the p-version and the hierarchical B-spline version of the Finite Cell Method for the geometrically nonlinear fictitious domain problem of a plate with a circular hole.

pressure boundary condition is taken into account by integration over \( \Gamma_N \). Fig. 23 shows an example of a hierarchical B-spline discretization, consisting of \( 12 \times 12 \) knot span elements of \( p=3 \) with \( m=3 \) hierarchical refinement levels, which adaptively resolve inner and outer boundaries \( \Gamma_N \) and \( \Gamma_D \). It also illustrates the approximation of the Neumann boundary \( \Gamma_N \) by polygon segments in the sense of Eq. 5.10.

Von Mises stresses plots, resulting from discretizations of \( 12 \times 12 \) knot span elements with no refinements and \( m=5 \) levels of adaptive hierarchical B-splines, are shown in Fig. 24. The contributions of the deformation dependent load to the external force vector and the stiffness matrix Eqs. (5.15) and (5.18) are obtained by integration over 940 and 1032 polygon segments in the unrefined and refined case, respectively. Due to the geometric nonlinearity of the system, the pressure load \( \hat{p} \) is divided into 20 equal increments, each of which requires 3 to 6 Newton-Raphson iterations to converge to a norm of the residual below \( 10^{-8} \). Whereas the B-spline discretization without hierarchical refinement yields a stress solution with strong oscillations over the complete domain, the adaptive refinement limits oscillations to the close vicinity of the geometric boundaries and leads to the correct circular stress pattern. This example illustrates that the beneficial behavior of the hierarchical B-spline FEM, which confines stress oscillations to the close vicinity of discontinuities, while leaving areas further away free of oscillations, also applies to the case of non-conforming boundary conditions.

7.4. Cube with spherical inclusion

The generalization of the 2D plate with circular inclusion to 3 dimensions leads to a cube with a spherical inclusion. A system sketch together with geometry, material and boundary conditions are given in Fig. 16b. For the computation...
of the corresponding reference solution, the symmetry in geometry and boundary conditions is used to reduce the cube to 1/8 of the original system. The considered FlagShyp discretization with a mesh conforming to the interface consists of 24,960 standard quadratic tetrahedral elements with 107,181 degrees of freedom. It yields a reference strain energy for the 1/8 system of $U_{ex} = 1.14299 \cdot 10^{-2}$. Convergence studies with different mesh sizes indicate that the given $U_{ex}$ is correct up to the 4th decimal, so that relative errors in strain energy in terms of Eq. 3.15, which are larger than 1%, can be reliably determined.

For the hierarchical B-spline computations, the origin of the coordinate system is again placed in the center of the inclusion, so that its geometry can be described implicitly by $X^2 + Y^2 + Z^2 \leq r^2$. The geometrically nonlinear system is solved by applying 5 equally sized load increments, each of which requires 3 to 5 Newton increments to converge to a norm of the residual below $10^{-12}$. In order to be able to compute higher resolutions with the computational power provided by a standard workstation, the hierarchical B-spline computations are also performed for the symmetrically reduced 1/8 system. The effect of hierarchical refinement in 3D is shown in Fig. 25, which compares Von Mises stress for an unrefined B-spline discretization of $5 \times 5 \times 5$ knot span elements of polynomial degree $p=3$ to a corresponding discretization with $m=3$ adaptive hierarchical refinement levels around the interface (see Fig. 25b).

In analogy to the 1D and 2D cases, the unrefined B-spline solution of Fig. 25c exhibits large oscillations throughout the domain, whereas the hierarchically refined B-spline solution of Fig. 25d confines oscillatory stresses to the close vicinity of the interface, while areas further away are free of oscillations. The refined B-spline solution is furthermore able to accurately localize the typical stress concentration phenomenon along the vertical edge of the interface.

Convergence in strain energy for the 3D benchmark is examined in Fig. 26. First, global $p$-refinement without local refinement is examined by increasing the polynomial degree from $p=3$ to 9 on a discretization of $p \times p \times p$ unrefined knot span elements. Due to the impact of oscillations, convergence in strain energy only achieves a rate of $q \approx 0.22$. Second, global $p$-refinement on the same grid is supplemented by a constant number of $m=2$ hierarchical refinement levels around the interface, which results in a reduced level of error at a slightly improved rate of convergence $q \approx 0.30$. Third, the effect of local $h$-refinement with hierarchical B-splines is examined, which starts from an unrefined B-spline discretization of $5 \times 5 \times 5$ knot span elements of constant polynomial degree $p=3$, adding one level of adaptive hierarchical B-splines per refinement step up to a maximum of $m=3$. Local $h$-refinement with hierarchical B-splines results in a rate of convergence of $q \approx 0.31$. The $hp$-strategy proposed in the previous sections is not examined here, since several hierarchical refinement levels at a higher polynomial degree $p$ in 3D imply an excessive increase in the bandwidth of the stiffness matrix, so that its solution requirements with appropriate direct solvers (Lapack, 2010; Pardiso, 2010) quickly exceed the hardware resources of standard workstations. Nonetheless, the convergence results illustrate that hierarchical refinement is able to reduce the error level and to improve the rate of convergence also in the 3D case. Hierarchical B-spline discretizations of moderate polynomial degree $p$ with local hierarchical refinement around interfaces offer a good trade-off between accuracy and accessibility with workstations.
7.5. Aluminium foam embedded in a rubber matrix

Multi-material composites provide high stiffness at reduced weights, and are therefore frequently used for lightweight structures in automotive and aerospace applications. A recent example are inter-penetrating phase composites (IPC), which combine open-cell aluminium foams with a filling material, such as rubber silicates (Chen & Han, 2003), epoxy-based syntactic foams (Jhaver & Tippur, 2009) or thermoplastic polymers (Cluff & Esmaeili, 2009).

The hierarchical B-spline FEM is applied to simulate a nonlinear compression test of an IPC sample, whose geometry is given by the spatial voxel model introduced in Section 6.2. As shown for the example of the 3D cube with spherical inclusion, local hierarchical $h$-refinement of interfaces starting from a grid of moderate polynomial degree can be expected to yield good accuracy at a manageable computational cost. Therefore, the foam sample of dimensions $10 \times 10 \times 10$ mm is discretized by a cubic B-spline grid of $8 \times 8 \times 8$ unrefined knot span elements, which is adaptively refined around material interfaces by 3 levels of hierarchical B-splines, amounting to a system with 215,799 degrees of freedom. The corresponding hierarchical grid structure resulting from fast grid generation as discussed in Section 6.2 is shown in Figs. 14 and 15. The IPC sample is assumed as part of a larger specimen, which is uniformly compressed along the vertical axis. A corresponding RVE model (Sehlhorst et al., 2009) specifies Dirichlet boundary conditions as follows: Displacements normal to the top surface are gradually increased to -1 mm (10% compressive deformation), modelling the influence of a testing machine, whereas the displacements normal to all other surfaces are fixed due to the bottom support and the influence of the surrounding material of the specimen. The aluminium foam is characterized by Young’s modulus $E_{\text{foam}}=70,000$ N/mm² and Poisson’s ratio $\nu = 0.35$. The material parameters of the embedding alkyd resin matrix are $E_{\text{matrix}}=21,000$ N/mm² and $\nu = 0.35$. The geometrically nonlinear system is solved.
by applying the displacement load on top in 4 increments, each of which requires 5 to 6 Newton-Raphson iterations to converge below a norm of the residual of $10^{-8}$.

Convergence with increasing adaptive hierarchical refinement is illustrated in Fig. 27 by the vertical top force, obtained by integrating the normal stress component over the top surface in B-spline discretizations with $m=0,1,2$ and 3 hierarchical refinement levels. It can be observed that the vertical top force converges to a value of around 451 kN. Von Mises stresses obtained with full hierarchical refinement of $m=3$ levels are plotted in Fig. 28a and illustrate that the hierarchical B-spline FEM effectively limits oscillations around the material interfaces and yields steady stress patterns free of oscillations away from the interfaces. Characteristic phenomena of multiphase materials typically occurring close to interfaces are stress concentrations in the stronger phase and stress minima in the weaker phase, which can be reproduced by the hierarchical B-spline discretization and are clearly observable in the stress plot. The present example demonstrates the potential of the hierarchical B-spline FEM for the simulation of multiphase structures with very complex interface geometries, where avoiding the expensive generation of explicit interface parameterizations or conforming meshes can considerably simplify and speed up the simulation process.
7.6. Open-cell aluminium foam

If the stiffness of the embedding matrix material is penalized by parameter $\alpha=10^{-4}$ in the sense of the fictitious domain approach as introduced in Section 4, the simulation reproduces the behavior of the open-cell aluminium foam sample without matrix filling. The resulting geometrically nonlinear fictitious domain problem is solved in the sense of the hierarchical B-spline version of the Finite Cell Method, using the same hierarchical B-spline discretization as in the interface case of the previous sub-section (see Figs. 14 and 15). Here, the displacement load is cut into 10 increments, each of which requires 5 to 8 Newton-Raphson iterations to converge to a residual with a norm below $10^{-6}$.

The convergence of the top force with increasing hierarchical refinement around geometric boundaries is shown in Fig. 27. The top force converges quickly to a value of around 150 kN. The Von Mises stresses obtained with full hierarchical refinement of $m=3$ levels are plotted in Fig. 28b. Stresses are free of large-scale oscillations and exhibit accurate localization of stress concentrations at the convex sides of the foam members, which agrees well with engineering experience. The open-cell aluminium example confirms again that the FCM version of the hierarchical B-spline FEM is able to extend the Finite Cell idea presented in Section 4 to geometrically nonlinear problems. Beside the direct simulation of complex foam structures without costly mesh generation, possible applications of the FCM version comprise homogenization and FE$^2$ approaches for repeated nonlinear RVE computations (see for example Zohdi & Wriggers (2005) or Sehlhorst et al. (2009)).

8. Summary and conclusions

In the present paper, the hierarchical B-spline FEM for interface problems is introduced, which circumvents a direct parameterization of the interface and offers fast and simple meshing irrespective of the geometric complexity involved. Its main idea is the combination of higher order approximation with the Gauss point method, which shifts the geometry description from the discretization to integration point level, and the unfitted adaptive refinement of interfaces by hierarchically contracted B-splines, which improves the resolution of corresponding discontinuities, but also allows to maintain a simple regular grid of knot span elements. The generalization of the hierarchical B-spline FEM for two- and three-dimensional problems is straightforward, since the simplicity of hierarchical refinement directly carries over to higher dimensions. Special strategies for the imposition of Dirichlet and deformation dependent Neumann boundary conditions along arbitrary curves were discussed.

The basic advantage of the regular grid approach is the fast and simple grid generation from an explicit representation of potentially very complex geometries. This was illustrated for a spatial voxel model of a foam composite sample, for which hierarchical grid generation was shown to be fast and easy to automate. The accuracy and computational efficiency of the hierarchical B-spline FEM were examined by a number of benchmark problems in one, two and three dimensions. Corresponding stress plots showed that stress oscillations around interfaces, typical for unfitted FE schemes, are restricted to the close vicinity of discontinuities and steady stress patterns are achieved in regions further
away from the interfaces despite the continuity of the B-spline patch. In particular, it was shown that the relative error in strain energy converges exponentially, when a \(hp\)-refinement strategy is applied, which simultaneously increases the number of local hierarchical refinement levels and the global polynomial degree \(p\) of the B-spline basis. Alternatively, discretizations of moderate polynomial degree with local hierarchical refinement around interfaces were found to be a good trade-off between accuracy and accessibility with standard hardware resources for expensive 3D computations. It was also demonstrated that the hierarchical B-spline FEM can be beneficially applied to geometrically nonlinear fictitious domain problems in a Finite Cell sense. In contrast to the \(p\)-version of the FCM, where nonlinear strain measures provoke large oscillations in stresses, which considerably affect the rate of convergence, the hierarchical B-spline version of the FCM can maintain high rates of convergence, since hierarchical refinement around geometric boundaries effectively limits oscillations, while preserving the key advantage of easy mesh generation. However, exponential rates of convergence could not be achieved in the FCM case.

The practical applicability of the hierarchical B-spline FEM was demonstrated by the nonlinear simulation of a compression test for an IPC foam composite and an aluminium foam sample, which could be analyzed on the basis of their voxelized geometry description without expensive mesh generation. Potential fields of application of the hierarchical B-spline FEM and the corresponding FCM version comprise for example the analysis of single- and multi-phase complex structures, homogenization methods and \(FE^2\) approaches.

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