Analytical Elastic Solution Based on Fourier Series for a Laterally Confined Granular Column

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Abstract

This paper presents an analytical solution methodology for the complete stress and displacement fields of a laterally confined granular column loaded from the top end. It is idealized as a homogeneous isotropic elastic medium with Coulomb’s friction at the lateral boundary. The solution methodology consists of an analytical procedure that incorporates a potential approach with trigonometric series and Bessel functions, finite Fourier transforms and the superposition method, and an iterative algorithm to satisfy the Coulomb’s friction condition at the lateral boundary. Stress and displacement fields are computed for a specific example and found completely consistent with corresponding finite element results. Key characteristics, computational errors, the convergence behavior and restrictions of the present approach are discussed. The methodology developed herein can be beneficially applied in the validation process of numerical simulation techniques in granular mechanics such as finite or discrete element methods.

KEYWORDS: Classical granular mechanics, Coulomb’s friction, Janssen’s silo model, K-value, Linear theory of elasticity, Potential trigonometric series approach with Bessel functions, Finite Fourier transforms.

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Introduction

The mechanical status of granular materials is still an open and much debated issue (Ooi et al. 2001; Herrmann et al. 2005). Traditional continuum models that have become especially sophisticated in soil mechanics (Kolymbas 2000; Yamamuro and Kaliakin 2005) idealize the discrete particles as a continuous medium with homogeneous mechanical parameters at the macroscopic level. At the mesoscopic level, however, the inhomogeneities in the spatial distribution of grain contacts lead to the formation of discrete force chains and arches (Duran 1999) that can hardly be modeled by a continuous medium. Within the last few decades, there has therefore been an intense discussion within the physics and engineering communities about the applicability and restrictions of the continuum approach in granular mechanics (see for example Savage 1998; Holst et al. 1999b; Herrmann et al. 2005). Several intensive research activities have also been devoted to the development of alternative theoretical approaches and numerical simulations for the determination of stresses and displacements in granular assemblies. Well known among those is the class of discrete element models (DEM) that take into account the behavior of any single element of a granular system [see for example Cundall and Strack (1979); Holst et al. (1999b); Ooi et al. (2001); Cook and Jensen (2002); and Ng (2004, 2006)].

A standard way of checking the validity of new computational granular models remains nonetheless the comparison of the results with traditional continuum elastic theories. Analytical solutions for elastic problems are limited to simple cases (Brown and Richards 1966; Drescher 1991; Nedderman 1992) and current research in granular elasticity is therefore largely focused on models within the numerical framework of the finite element method [see for example Lade and Nelson (1987); Chang et al. (1995); Taciroglu and Hjelmstad (2002); Chang and Shi (2005); Hicher and Chang (2006)]. An exception is the basic case of a cylindrical container filled with granular material as shown in Fig. 1, for which the analytical Janssen model exists. It is a popular choice for the validation of computational results in granular mechanics [see for example Holst et al. (1999a, 1999b); Chen et al. (2001); Landry et al.
(2003); Ovarlez and Clément (2005); Vidal et al. (2006)], because it is based on the familiar concept of linear elasticity, mathematically simple and thoroughly investigated.

The analytical derivation of the model, which dates back over 100 years to the pioneering paper by Janssen (1895), is based on two simplifying assumptions. First, the frictional shear stress $\sigma_{rz}$ between particles and container wall satisfies the Coulomb’s failure criterion, i.e. the particles are at the onset of sliding at any point of the wall. It is expressed as $\sigma_{rz} = \mu \sigma_{rr}$, where $\mu$ is the static coefficient of Coulomb’s friction. Second, the vertical and horizontal normal stresses, $\sigma_{zz}$ and $\sigma_{rr}$ respectively, are assumed to be principal stresses whose ratio $K$ is constant in the form $\sigma_{rr} = K \sigma_{zz}$. By performing a vertical force balance on a differential slice of material, a linear first-order differential equation is found in the form

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{2\mu K}{R} \sigma_{zz} = \rho g$$

(1)

where $\rho$ is the density, $g$ the acceleration due to gravity, and $R$ the radius of the cylinder (Nedderman 1992). The solution yields the well-known Janssen formulae

$$\sigma_{zz}^J = \frac{\rho g R}{2\mu K} \left[ 1 - \exp \left( -\frac{2\mu K}{R} z \right) \right] + \sigma_0 \exp \left( -\frac{2\mu K}{R} z \right)$$

(2)

$$\sigma_{rr}^J = \frac{\rho g R}{2\mu} \left[ 1 - \exp \left( -\frac{2\mu K}{R} z \right) \right] + K \sigma_0 \exp \left( -\frac{2\mu K}{R} z \right)$$

(3)

$$\sigma_{rz}^J = \frac{\rho g R}{2} \left[ 1 - \exp \left( -\frac{2\mu K}{R} z \right) \right] + \mu K \sigma_0 \exp \left( -\frac{2\mu K}{R} z \right)$$

(4)

with $\sigma_0$ being a constant stress surcharge at the top $z = 0$. The upper index “J” indicates throughout the paper that some stress quantity belongs to the standard Janssen model. Eqs. (2) and (3) are the vertical and horizontal normal stresses that are constant over each cross section. Eq. (4) represents the shear stress at the lateral boundary due to friction between grains and the container wall.
However, the Janssen model has considerable shortcomings. The basic K-value assumption violates the governing equations of elasticity (Nedderman 1992). It reduces the underlying system of partial differential equations to the simple ordinary differential equation Eq. (1), which yields only a one-dimensional stress solution for the three-dimensional problem shown in Fig. 1. Moreover, the Janssen model is not able to provide any information about displacements, since it is based solely on equilibrium considerations.

Against this background, this paper proposes an analytical solution methodology to obtain the complete elastic stress and displacement fields in axisymmetric cylindrical coordinates \( r \) and \( z \) for a laterally confined granular column. The solution fields satisfy the governing equations of linear elasticity and the boundary conditions of zero lateral confinement and constant top surcharge analytically. The boundary condition of Coulomb’s friction is accomplished numerically by an iterative algorithm. The results of the present approach converge towards the exact elastic solution for the laterally confined cylinder shown in Fig. 1.

First, the appropriate elastic boundary value problem is formulated and the method of superposition is briefly presented. An analytical solution procedure is then developed including a potential trigonometric series approach with Bessel functions and the application of finite Fourier transforms. Since the latter requires an explicit stress expression at the lateral boundary, the boundary condition of Coulomb’s friction is replaced by the known shear stress distribution of the Janssen model. The analytical solution procedure is then embedded into an iterative algorithm that ensures the satisfaction of the Coulomb’s condition at the lateral boundary. Finally, the analytical solution fields are completed by superposition of the influence of the weight of the material. Stress and displacement fields are computed for an example problem and compared to corresponding finite element results. Key characteristics of the analytical solution, the influence of computational errors and the convergence behavior of the iterative algorithm are discussed. In Appendix A, a code written for the mathematical software package MATLAB (2006) is given that can be used to determine stresses and displacements in confined columns with arbitrary geometric, elastic and frictional properties.
The situation of a laterally confined granular medium with wall friction is of substantial importance in many engineering applications, e.g. in caissons construction and pile driving, in the design of silos and storage bins or in chemical molding processes. Beyond that, the present analytical solution methodology could be used beneficially in the validation process for new theoretical and numerical approaches in granular mechanics, such as DEM based formulations or finite element computations, which its field solutions are better suited for than the standard one-dimensional Janssen results.

**Fundamental Analytical Principles**

The physical problem investigated in this study consists of a column of radius $R$ made up of discrete granular particles (Fig. 1). The granular medium is idealized as an isotropic, linear elastic continuum with homogeneously distributed mass density $\rho$, Young’s modulus $E$, and Poisson’s ratio $\nu$. It is confined by a rigid container wall, so that a lateral bulging of the material is completely prevented at the lateral surface $r = R$. At the top $z = 0$, the medium is loaded by a surcharge of constant stress $\sigma_0$. The column is assumed to be infinitely long in positive vertical direction $z$. The frictional mechanism between grains and container wall at the lateral boundary is given by the Coulomb’s friction condition $\sigma_{rz} = \mu \sigma_{rr}$.

**Governing Equations of Linear Elasticity and Boundary Conditions**

Mathematically, this situation is represented by the following boundary value problem that consists of the governing equations of linear elasticity and the boundary conditions. Since both geometry and loads in Fig. 1 are symmetrical with respect to the vertical axis $z$, the governing equations in cylindrical coordinates $(r, \theta, z)$ are independent of $\theta$ and can thus be formulated for the axisymmetric case as follows (Timoshenko and Goodier 1970).
Strain-Displacement Relations:

\[
\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \quad \varepsilon_{\theta\theta} = \frac{u_\theta}{r} \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \quad \varepsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \]

\[
2 \varepsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \quad \varepsilon_{r\theta} = \varepsilon_{z\theta} = 0
\]

Equilibrium Equations:

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{\theta\theta}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \quad \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{\sigma_{r\theta}}{r} - \rho g = 0
\]

Constitutive Relations:

\[
\varepsilon_{rr} = \frac{1}{E} [\sigma_{rr} - \nu (\sigma_{\theta\theta} + \sigma_{zz})] \quad \varepsilon_{\theta\theta} = \frac{1}{E} [\sigma_{\theta\theta} - \nu (\sigma_{rr} + \sigma_{zz})] \quad \varepsilon_{zz} = \frac{1}{E} [\sigma_{zz} - \nu (\sigma_{rr} + \sigma_{\theta\theta})] \quad \varepsilon_{rz} = \frac{(1 + \nu)}{E} \sigma_{rz}
\]

The quantities \( u \) denote the displacements, \( \varepsilon \) the strains, and \( \sigma \) the stresses, along the directions of their subscripts. Eqs. (5) through (7) guarantee that equilibrium of stresses and compatibility of strains are satisfied for an arbitrary element of the elastic cylinder.

The set of boundary conditions according to Fig. 1 are

At \( z = 0 \):

\[
\sigma_{zz}(r) = \sigma_0 \quad (8)
\]

At \( r = R \):

\[
u_r(z) = 0 \quad (9)
\]

\[
\sigma_{rz} = \mu \sigma_{rr} \quad (10)
\]

In this paper, positive stresses and displacements indicate compression, following the established sign convention used in granular mechanics. Since the granular bed is considered to be semi-infinitely long, no direct boundary condition in terms of the vertical displacements can be formulated at a finite length. Therefore, the vertical displacements are determined relative to a point on the lateral surface at some arbitrary depth \( Z \).

At \( z = Z \) and \( r = R \):

\[
u_z = 0 \quad (11)
\]
Eq. (11) does not correspond to the actual displacements of the physical problem, but serves only as a reference value. The set of Eqns. (5) through (11) constitute the elastic boundary value problem for a laterally confined continuum elastic column, whose analytical solution, so far to the authors’ knowledge, has not yet been found.

The Method of Superposition

In this paper, an analytical solution approach for the elastic boundary value problem Eqs. (5) through (11) is presented. The superposition of stresses and displacements is allowed in linear elasticity (Timoshenko and Goodier 1970). The final elastic stress and displacement solution \( S \) consists of three independent solution components \( S^I, S^{II}, \) and \( S^{III} \) to be superposed during the solution process.

\[
S = S^I + S^{II} + S^{III} \tag{12}
\]

Each term in Eq. (12) can be related to a particular stress scenario. The first component \( S^I \) corresponds to a confined stress state of uniform compression due to the top surcharge \( \sigma_0 \) without considering the material weight and the friction at the wall. The second solution component \( S^{II} \) represents stresses generated only by the wall friction resulting from the top surcharge \( \sigma_0 \), but without considering the surcharge itself. The third component \( S^{III} \) incorporates the contribution due to the material weight represented by the mass density \( \rho \).

Analytical Solution Procedure

Many researchers have been involved in the development of analytical models with the target of approximating the exact solution of the elastic boundary value problem Eqs. (5)
through (11) as accurately as possible [see for example Janssen (1895); Jaky (1948); Lenczner (1963); Walker (1966); Horne and Nedderman (1976); Cowin (1977); Wittmer et al. (1997); Zhang et al. (1998); Malla and Anandakumar (2004, 2006); Ovarlez and Clément (2005); Schillinger and Malla (2007)]. These models are based on simplifying assumptions - such as the K-value assumption in the Janssen model - that release the strict conditions given by the governing equations of elasticity, but satisfy the boundary conditions exactly.

**Explicit Shear Stress at the Lateral Boundary**

In this paper, the problem is approached in a different way. The governing equations of elasticity are strictly satisfied, but one of the boundary conditions, namely the Coulomb’s condition Eq. (10), is replaced by a known shear stress distribution along the lateral surface. This becomes necessary as the analytical solution procedure is based on the finite Fourier transform technique that asks for an explicit stress expression at the lateral boundary.

The exact elastic wall shear stress inherently satisfying the Coulomb’s condition is of course not known a-priori. However, various approximations can be obtained from the approximating analytical models cited above. In the case presented here, it is decided to choose the shear stress distribution of the Janssen model Eq. (4) for the weightless case.

\[
\sigma_{rz}^{J} = \mu K \sigma_0 \exp\left(-\frac{2\mu K}{R} z\right)
\]  

At \( r = R \):

\[
\sigma_{rz}^{J} = \mu K \sigma_0 \exp\left(-\frac{2\mu K}{R} z\right)
\]  

(13)

Since the influence of the weight of the material will be superposed in closed form later, the present analytical solution procedure assumes \( \rho = 0 \) for the moment.

Due to the new boundary condition Eq. (13) based on the simplifying assumption of Janssen’s constant K-value, the satisfaction of the Coulomb’s condition at the wall is not generally ensured. The analytical solution procedure will be therefore embedded into an algorithm that uses the Janssen shear stress Eq. (13) as a starting point to iteratively approach the exact elastic solution, which satisfies the Coulomb’s condition at the wall. The derivation
of the analytical solution procedure is presented in the following, while the derivation of the iterative algorithm will be the subject of the section to follow.

**Stress and displacement fields for uniform compression**

Due to \( \rho = 0 \) for the moment, the analytical solution procedure deals only with solution components \( S^I \) and \( S^{II} \). The first component \( S^I \) corresponds to a confined stress state of uniform compression without friction at the wall and covers the vertical stress surcharge \( \sigma_0 \) at the top. It consists of the stress and displacement fields for a confined elastic cylinder under uniform vertical compression without friction at the lateral surface. This is a regular scenario that is widely discussed for example by Timoshenko and Goodier (1970). The corresponding stress and displacement expressions of \( S^I \) are

\[
\begin{align*}
\sigma_n^I (r, z) &= \sigma_{yy}^I (r, z) = \frac{\nu}{1-\nu} \sigma_{zz}^I (r, z) \quad (14a) \quad \sigma_{zz}^I (r, z) &= \sigma_0 \quad (14b) \\
\sigma_{zz}^I (r, z) &= 0 \quad (14c) \quad u^I_n (r, z) = 0 \quad (14d) \\
u^I_z (r, z) &= \sigma_0 \frac{1+\nu}{E} \left( \frac{2\nu-1}{\nu-1} \right) + u_0 \quad (14e)
\end{align*}
\]

where \( u_0 \) is an arbitrary constant. The validity of Eqs. (14a-e) with respect to linear elasticity can be checked by substitution into the governing equations (5) through (7). Boundary conditions Eq. (8) at the top and Eq. (9) at the lateral surface are satisfied. However, the Janssen shear stress distribution Eq. (13) is not yet accomplished.

**Potential Approach for Frictional Shear**

The second solution component \( S^{II} \) represents the frictional shear stress reaction at the lateral boundary due to the vertical surcharge \( \sigma_{zz}^I \) of Eq. (14b). The stress and displacement fields of \( S^{II} \) are derived from a potential approach whose derivation and various other applications are described extensively in previous contributions [see Love (1944); Timoshenko...
Introducing a biharmonic function $\Psi(r,z)$ such that

$$\sigma_{rr} = \frac{\partial}{\partial z} \left[ \nu \nabla^2 \Psi - \frac{1}{r} \frac{\partial \Psi}{\partial r} \right]$$

(15a)

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z} \left[ \nu \nabla^2 \Psi - \frac{1}{r} \frac{\partial \Psi}{\partial r} \right]$$

(15b)

$$\sigma_{z\theta} = \frac{\partial}{\partial r} \left[ (2-\nu) \nabla^2 \Psi - \frac{\partial^2 \Psi}{\partial z^2} \right]$$

(15c)

$$\sigma_{zz} = \frac{\partial}{\partial r} \left[ (1-\nu) \nabla^2 \Psi - \frac{\partial^2 \Psi}{\partial z^2} \right]$$

(15d)

$$u_r = \frac{1 + \nu}{E} \frac{\partial^2 \Psi}{\partial r \partial z}$$

(15e)

$$u_z = \frac{1 + \nu}{E} \left[ 2 (1-\nu) \nabla^2 \Psi - \frac{\partial^2 \Psi}{\partial z^2} \right]$$

(15f)

the governing equations (5) through (7) are satisfied, provided

$$\nabla^2 \nabla^2 \Psi = 0$$

(16)

where $\nabla^2$ is the axisymmetric Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

(17)

It can be verified by substitution that Eq. (16) has a trigonometric series solution

$$\Psi = \sum_{n=1}^{\infty} \cos(\alpha_n z) \left[ A_n I_0(\alpha_n r) + B_n \alpha_n r I_1(\alpha_n r) \right]$$

(18)

where $I_k(\alpha_n r)$ denote modified Bessel functions of the kind $k$. $\alpha_n$, $A_n$ and $B_n$ are sets of constants to be determined from the lateral boundary conditions Eqs. (9) and (13). Substitution of Eq. (18) into Eqs. (15a-f) yields the following stress and displacement expressions

$$\sigma_{rr}(r,z) = \sum_{n=1}^{\infty} \alpha_n^3 \sin(\alpha_n z) \left\{ A_n \left[ I_0(\alpha_n r) - \frac{I_1(\alpha_n r)}{\alpha_n r} \right] + B_n \left[ (1-2\nu)I_0(\alpha_n r) + \alpha_n r I_1(\alpha_n r) \right] \right\}$$

(19)
\[ \sigma^H_{zz}(r, z) = \sum_{n=1}^{\infty} \frac{\alpha_n^{2}}{r} \sin(\alpha_n z) \left\{ A_n I_1(\alpha_n r) + B_n \alpha_n \alpha_n (1 - 2\nu) I_0(\alpha_n r) \right\} \]  

\[ \sigma^H_{rz}(r, z) = \sum_{n=1}^{\infty} -\alpha_n^{3} \sin(\alpha_n z) \left\{ A_n I_0(\alpha_n r) + B_n \left[ \alpha_n r I_1(\alpha_n r) + 2(2 - \nu) I_0(\alpha_n r) \right] \right\} \]  

\[ \sigma^H_{zz}(r, z) = \sum_{n=1}^{\infty} \alpha_n^{3} \cos(\alpha_n z) \left\{ A_n I_1(\alpha_n r) + B_n \left[ \alpha_n r I_0(\alpha_n r) + 2(1 - \nu) I_1(\alpha_n r) \right] \right\} \]  

\[ u^H_z(r, z) = \frac{(1 + \nu)}{E} \sum_{n=1}^{\infty} \sin(\alpha_n z) \left[ A_n \alpha_n^{2} I_1(\alpha_n r) + \alpha_n^{3} r B_n I_0(\alpha_n r) \right] \]  

\[ u^H_z(r, z) = \frac{(1 + \nu)}{E} \sum_{n=1}^{\infty} \cos(\alpha_n z) \left[ A_n \alpha_n^{2} I_0(\alpha_n r) \right. \]  

\[ \quad + B_n \left\{ 4 (1 - \nu) \alpha_n^{2} I_0(\alpha_n r) + \alpha_n^{3} r I_1(\alpha_n r) \right\} \]  

\[ + u_0 \]  

where \( u_0 \) is again an arbitrary constant.

The main objective here is to determine the three sets of constants \( \alpha_n, A_n, \) and \( B_n \) in such a way that the stress and displacement fields of solution component \( S^H \) satisfy the boundary condition Eq. (13) without interfering in the superposition \( S^I + S^H \) with the boundary conditions of constant top surcharge Eq. (8) and zero lateral displacements Eq. (9) that are already satisfied in component \( S^I \). Therefore, the factor \( \cos(\alpha_n z) \) is chosen in the potential Eq. (18), because this results in the factor \( \sin(\alpha_n z) \) in Eq. (21). The sine factor renders the vertical stresses automatically zero for \( z = 0 \) thus preventing an influence of Eq. (21) on the top boundary condition Eq. (8).

Another implication of the cosine factor in Eq. (18) is the fact that the horizontal displacements are constrained to be zero due to the sine factor in Eq. (23). However, this mathematical constraint can be readily justified from a physical point of view. In experiments and applications, the cylinder is loaded by a rigid piston that is able to provide resistance against horizontal displacements at the top surface thus equilibrating the resulting shear stresses by friction. Therefore, this constraint is accepted here.
Finite Fourier Cosine Transforms

The constants $A_n$ and $B_n$ are now determined by means of Eigenfunction expansion that mainly bases on finite Fourier transforms (Kreyszig 2005). The lateral confinement Eq. (9) can only be satisfied in general, if the bracket term in Eq. (23) is zero for $r = R$, which yields

$$A_n = -B_n \frac{\alpha_n R I_1(\alpha_n R)}{I_1(\alpha_n R)}$$

(25)

Now the set of Eigenvalues $\alpha_n$ is chosen from the characteristic equation Eq. (26) in such a way that the cosine series constitutes an orthogonal set on the sampling interval $z \in [0; L]$ (Kreyszig 2005).

$$\cos(\alpha_n L) = 0 \quad \rightarrow \quad \alpha_n = \frac{(2n - 1) \pi}{2L} \quad (n = 1, 2, ..., \infty)$$

(26)

The sampling length $L$ has to be finite and is therefore not able to cover the whole length of the semi-infinite cylinder. But it can be extended arbitrarily, so that the solution fields for any top portion $z \in [0; L]$ can be found. If Eq. (25) is substituted into Eq. (22), evaluated at $r = R$, it follows that

$$\sigma_{rr}^H(R, z) = \sum_{n=1}^{\infty} B_n \alpha_n^3 \cos(\alpha_n z) \cdot \{2(1 - \nu) I_1(\alpha_n r)\}$$

(27)

The set of constants $B_n$ can now be determined from Eq. (27) by finite Fourier cosine transforms as follows. Substitute Eq. (13) into the left side of Eq. (27), multiply the resulting expression by $\cos(\alpha_n z)$ and integrate it over the interval $[-L; L]$. This yields the relation

$$\int_{-L}^{L} \mu K \sigma_0 \exp\left(-\frac{2\mu K}{R} z\right) \cdot \cos(\alpha_m z) \, dz = \sum_{n=1}^{\infty} B_n \alpha_n^3 \{2(1 - \nu) I_1(\alpha_n r)\} \cos(\alpha_m z) \, dz$$

(28)
where the index \( m = \{1, 2, 3, \ldots, \infty\} \) is fixed. According to Eq. (26), the power series on the right side of Eq. (28) constitutes an infinite set of orthogonal cosine functions, for which the following relations hold

\[
\int_{-L}^{L} \cos(\alpha_n z) \cdot \cos(\alpha_m z) \, dz = \begin{cases} 
L & \text{if } n = m \\
0 & \text{if } n \neq m
\end{cases} \tag{29}
\]

Hence, only the \( m \)th term remains in the power series of Eq. (28). Because of the symmetry of the cosine function, the integration interval can be reduced to \([0; L]\) and the integrand is multiplied by two. Thus, Eq. (28) can be explicitly solved for \( B_m \) as

\[
B_m = \frac{\mu K \sigma_0}{\alpha_m^3 \{2(1-\nu) I_1(\alpha_m r)\} L} \cdot \int_{0}^{L} 2 \exp\left(-\frac{2\mu K}{R} z\right) \cdot \cos(\alpha_m z) \, dz \tag{30}
\]

Fourier transforms can be applied consecutively in Eq. (30) for any \( m = \{1, 2, 3, \ldots, \infty\} \). However, the series can only be evaluated for a finite number of \( m \) and must therefore be truncated at some point. The corresponding \( A_m \) can be easily computed from Eq. (25).

At this point, the incompatibility of the Coulomb’s condition to the application of finite Fourier transforms becomes obvious. If Eqs. (19) and (22) are substituted into the Coulomb’s condition \( \sigma_{rz}^{\mu} = \mu \sigma_{rr}^{\mu} \), there is a \( \cos(\alpha_n z) \) on the left side of this expression, for which a Fourier cosine transform would be needed to isolate \( B_m \), and a \( \sin(\alpha_n z) \) on the right side, for which a sine transform would be needed. Since only one kind of Fourier transform can be applied at a time, the Coulomb’s condition is incompatible to Fourier transforms.

The analytical solution \( S^{\mu} \) can be found by substitution of \( \alpha_n \), \( A_n \) and \( B_n \) into Eqs. (19) through (24) and the elastic solution \( S^I + S^{\mu} \) can then be assembled by superposition. It satisfies both the governing equations of elasticity Eqs. (5) through (7) and the boundary
conditions Eqs. (8), (9) and (13). Boundary condition Eq. (11) can be easily satisfied by adjustment of the arbitrary constants \( u_0 \) in Eqs. (14e) and (24).

**Iterative Solution Algorithm**

The solution fields resulting from the analytical solution procedure alone do not generally satisfy the Coulomb’s friction condition Eq. (10). The analytical solution procedure is therefore embedded into an algorithm that iteratively approaches the solution with exact Coulomb’s condition at the lateral boundary. It is based on the derivation of a general K-function \( K(z) \) that is updated after each iteration starting from the initial Janssen’s constant K-value.

**Determination of the K-function**

The upper index “\( l+II \)” indicates in the following that some stress quantity belongs to the analytical solution containing the superposition of components \( S^I + S^{II} \), which has been obtained in the previous section. Consider the stresses acting on a cylindrical slice of thickness \( dz \) at some arbitrary depth \( z \) of the elastic medium as shown in Fig. 2. Due to boundary condition Eq. (13), the Janssen shear stress \( J_{rz} \) and the shear stress \( J_{rz}^{l+II} \) are equivalent. Vertical equilibrium confirms that the change in vertical force \( \frac{\partial P}{\partial z} \) is equal to the shear stress integrated over the circumference of the slice. The vertical force \( P \) can be equally obtained by integration of the vertical stress \( J_{zz}^{l} \) (solid line) or \( J_{zz}^{l+II} \) (dashed line) over the cross section. Hence, \( P/A \) corresponds equivalently to the constant vertical stress \( J_{zz}^{l} \) and the averaged stress \( J_{zz}^{l+II} \). This can be summarized as

\[
\sigma_{zz}^{l}(z) = \frac{1}{A} \int_0^{2\pi} \int_0^R J_{zz}^{l+II}(r,z) r \, dr \, d\theta = \frac{2}{R} \int_0^R J_{zz}^{l+II}(r,z) r \, dr \quad (31)
\]
with $A = \pi R^2$ being the cross sectional area of the cylinder.

By substitution of Eqs. (2) and (31) into Eq. (13), a direct relation between the shear stress at the lateral boundary $\sigma_{rz}^{I+II}(R)$ and the vertical stress $\sigma_{zz}^{I+II}$ averaged over the cross section can be found in the form

$$\sigma_{rz}^{I+II}(R) = \mu K \sigma_{zz}^J = \frac{2 \mu K}{R^2} \int_0^R \sigma_{zz}^{I+II} r \, dr$$

(32)

The Coulomb’s condition at the lateral surface $\sigma_{rz}^{I+II}(R) = \mu \sigma_{rr}^{I+II}$ can be substituted into Eq. (32) and the resulting expression is solved for $K$.

$$K(z) = \frac{\sigma_{rz}^{I+II}(R)}{\frac{2}{R^2} \int_0^R \sigma_{zz}^{I+II} r \, dr}$$

(33)

The resulting $K$-function Eq. (33) being dependent on $z$ clearly differs from the initial constant $K$ that has been defined as Janssen’s constant $K = \sigma_{rr}^J / \sigma_{zz}^J$.

**Theoretical Concept of the Algorithm**

There exists an unknown function $K(z)$, which leads to the exact satisfaction of the Coulomb’s condition when used in Eq. (13) of the analytical solution procedure. It is now postulated that the K-function of Eq. (33) is closer to this exact $K(z)$ than the previous constant K-value. If the analytical solution procedure is re-run using the new K-function Eq. (33) in the explicit shear stress expression Eq. (13), it can be accordingly expected that the new solution satisfies the Coulomb’s condition at the wall better than the previous one.

On this basis, an iterative algorithm that is capable of determining iteratively the exact solution fields of the boundary value problem Eqs. (5) through (11) can be constructed as follows. Starting from the analytical solution presented in the previous section, the K-function Eq. (33) is evaluated and inserted into the explicit shear stress Eq. (13), which is in turn used
to re-run the analytical solution procedure. The resulting solution fields can then be used to re-start the iteration process by evaluating Eq. (13) again. The convergence can be checked after each step by evaluation of the Coulomb’s condition. The important benefit of the iterative algorithm in this form is the preservation of the analytical nature of the solution, since the analytical solution process is repeated in each iteration.

It should be noted at this point that the new solution cannot be obtained by simply plugging $K(z)$ into Eq. (13), because the underlying Eq. (1) turns to an ordinary differential equation with non-constant coefficients with the general solution

$$\sigma_z = \mu K(z) \sigma_0 \exp\left(-\frac{2\mu}{R} \int_0^z K(z) \, dz\right)$$

Eq. (34) can be obtained for example by separation of variables (Kreyszig 2005) and contains Eq. (13) for the special case of constant $K$.

### The Complete Solution

The analytical solution fields are now completed by superposing the third solution component $S^{iii}$ to components $S^I + S^{ii}$ that have been previously obtained from the iterative algorithm. Component $S^{iii}$ representing the influence of the weight of the material is given as a set of closed form expressions. For practical application, the present analytical methodology needs to be implemented into a computational framework that is outlined briefly.

### Solution with Weight of the Material

In the limit of very high cylinder depths $z \to \infty$, the stresses of $S^I + S^{ii}$ decay to zero due to the exponential term in the lateral boundary condition Eq. (13). Therefore, the corresponding asymptotic stresses and displacements for large $z$ must be due to the weight of the material.
\( \rho g \) only (Nedderman 1992). A closed-form analytical solution for these asymptotic stresses and displacements is provided by Ovarlez and Clément (2005) and reads

\[
\sigma^{iii}_{rr}(r,z) = \sigma^{iii}_{\theta \theta}(r,z) = \frac{V}{1-\nu} \sigma^{iii}(r,z) = \frac{(1-\nu)\rho g R}{2 \nu \mu} \tag{35a}
\]

\[
\sigma^{iii}_{rz}(r,z) = \frac{1}{2} \rho g r \tag{35c}
\]

\[
u^{iii}(r,z) = 0 \tag{35d}
\]

\[
u^{iii}(r,z) = \frac{1 + \nu}{2E} \rho g r^2 + \frac{1 - \nu - 2 \nu^2}{2 \nu \mu E} \rho g R z + u_0 \tag{35e}
\]

with \( u_0 \) being an arbitrary constant to meet Eq. (11). Eqs. (35a-e) constitute the third solution component \( S^{iii} \) accomplishing the governing equations Eqs. (5) to (7) and the boundary conditions Eq. (9) and (10) exactly. Thus, the complete superposition \( S^I + S^{ii} + S^{iii} \) satisfies the governing equations Eqs. (5) through (7) and the boundary conditions of zero lateral displacement Eq. (9), Coulomb’s condition Eq. (10) and vertical displacements Eq. (11).

What remains is the accomplishment of boundary condition Eq. (8) representing the top surcharge \( \sigma_0 \). At the top surface \( z = 0 \), the vertical stress due to the weight of the material Eq. (35b) adds to the top surcharge \( \sigma_0 \) that is exactly satisfied in components \( S^I + S^{ii} \). Exceeding \( \sigma_0 \) can be prevented if \( \sigma_0 \) is replaced in the solution components \( S^I + S^{ii} \), in particular in the shear stress Eq. (13) and the uniform stress state Eqs. (14a-e), by the surcharge \( \hat{\sigma}_0 \) that has been reduced a-priori by the amount of the vertical stress of Eq. (35b).

\[
\hat{\sigma}_0 = \sigma_0 - \frac{(1-\nu)\rho g R}{2 \nu \mu} \tag{36}
\]

Thus, the vertical normal stress at \( z = 0 \) in the solution \( S^I + S^{ii} \) corresponds to \( \hat{\sigma}_0 \), so that after the superposition with Eqs. (35b) the desired \( \sigma_0 \) of Eq. (8) is accomplished exactly.
It is worthwhile to note here that for the special case of zero surcharge $\sigma_0 = 0$ (i.e. when the loading consists of only the weight of the material), the solution does not consist of component $S^{iii}$ alone; but all three solution components, i.e. $S^{iii}$ superimposed with components $S^{i}$ and $S^{ii}$ obtained considering $\sigma_0 = 0$ in Eq. (32). As can be readily seen, this is necessary to satisfy the relevant boundary condition of zero vertical stress at the free surface $z = 0$ for this loading case. The present solution methodology is also equally valid for the case of a tensile surcharge stress $\sigma_0 < 0$. However, in reality this stress scenario leads to the failure of most granular materials, since they are only able to carry compressive stresses. It has to be kept in mind that representation of failure due to tension is beyond the capability of the presented elastic model.

**Implementation of the Solution Procedure**

The present solution methodology can only be practically applied with the help of a computer, since the series representation and the finite Fourier transforms require computational evaluation. In this paper, the software package MATLAB (2006) is used. The corresponding code that can be easily adjusted to arbitrary cylinder problems can be found in Appendix A. Since the programming is straightforward, only a few peculiarities are mentioned here.

The solution fields are not stored as functions, but by their values at a finite number of discrete points in the cylinder domain $[r = 0...R; z = 0...L]$, which requires a discretization of the axisymmetric plane in $r$- and $z$-directions. Integration in the Fourier transforms must thus be carried out numerically. During each iteration, the K-function $K(z)$ is updated by evaluation of Eq. (33) at each discrete point in $z$-direction. The integration of the updated $K(z)$, which is required by Eq. (34), is done by approximating the values of $K(z)$ by splines (piecewise polynomial interpolation), which then can be integrated analytically from $z = 0$ to each discrete point $z$ (de Boor 1978).
Application, Results and Discussion

The analytical solution methodology developed in the foregoing sections is applied to a specific example from the authors’ research activities on water-processing beds for the application in space life support systems (Malla and Anandakumar 2004; Malla and Schillinger 2007). The parameter values used in the following pertain to activated alumina granulate confined in a steel cylinder, which are \( R = 37.5 \, mm \), \( Z = 150 \, mm \), \( L = 600 \, mm \), \( E = 15 \, N/\, mm^2 \), \( \nu = 0.3 \), \( \rho g = 0.0025 \, N/\, mm^3 \), \( \mu = 0.36 \) and \( \sigma_0 = 1 \, N/\, mm^2 \). The initial Janssen’s K-value is chosen as the elastic redirection coefficient \( K = \nu / (1 - \nu) = 0.4286 \) (Ovarlez and Clément 2005). The infinite series representation is truncated after \( n = 200 \) terms. Stresses and displacements computed with the MATLAB routine from Appendix A are graphically illustrated in Figs. 3 through 8. The corresponding Coulomb’s coefficient \( \mu \) at the lateral boundary representing boundary condition Eq. (10) is shown in Fig. 9.

Comparison with FEM Results

Stresses and displacements of the laterally confined elastic cylinder are obtained numerically by finite element computations with the commercial code ABAQUS (2004). The finite element mesh that consists of axisymmetric quadrilateral elements with a spatial resolution of 20/640 elements in horizontal and vertical direction, respectively, is schematically shown in Fig. 10. Geometry, top surcharge and material parameters correspond to the example from the beginning of this section. The top surface of the FE model is supported in horizontal direction to account for the displacement constraint introduced by the potential function Eq. (18). Additionally, the FE model has to be held in \( z \)-direction at a finite depth. A rigid vertical support is chosen at \( L = 1200 \, mm \) “far away from the top”, so that its influence on the considered part of the cylinder \( z = [0, 150 \, mm] \) is practically zero. At the lateral boundary the Coulomb’s condition is numerically incorporated by means of the penalty method (Bathe 1996).
The FE results represent a numerical solution for the elastic boundary value problem Eqs. (5) through (11). They are shown along with those obtained from the analytical solution fields in Figs. 3 through 8. It can be observed that the general mechanical behavior of the analytical results is completely consistent with the FE analysis. Both stress and displacement fields show excellent qualitative accordance. The FE results, however, do not accomplish the Coulomb’s friction condition at the wall as accurately as the analytical solution (see Fig. 9).

**Comparison with Janssen Results**

The analytical elastic solution and the analytical Janssen solution Eqs. (2) to (4) cannot be directly compared, because the former consists of solution fields varying over the radius and the latter constitutes a one-dimensional solution giving only one value at each depth. However, if the Janssen shear stress Eq. (2) is identified with the shear stress at the wall of the confined elastic medium, it can be directly compared to the analytical shear stress at \( r = R \).

Moreover, if the Janssen vertical stress Eq. (4) is related to the elastic vertical stress averaged over the radius, which can be inferred on the basis of Eq. (31), it can be directly compared to the analytical vertical stress averaged over \([0; R]\).

The Janssen shear and vertical stresses and the analytical shear and averaged vertical stresses, respectively, are evaluated for three different load cases: (1) material weight only, (2) top surcharge only and (3) material weight and top surcharge together. All parameters correspond again to the example problem described at the beginning of this section. The Janssen solution (dotted curves) and the analytical results (solid curves) are plotted for the shear and vertical stresses in Figs. 11 and 12, respectively. Due to the marginally small differences between analytical and Janssen results (below 2.5% maximum relative difference in any case), the Janssen shear and vertical stresses can be seen as very good approximations to the corresponding stresses of the analytical elastic solution. Furthermore, Figs. 11 and 12 clearly show that the differences decrease with depth, because in the asymptotic
case of \( z \to \infty \), the stress quantities of the Janssen model Eq. (2) to (4) and the asymptotic analytical solution represented by solution component \( S^{III} \) are identical.

**Trigonometric Nature of the Solution**

The potential approach in the extended Janssen model generates solution fields whose nature is periodic in \( z \)-direction, since their behavior is largely determined by sines or cosines - see Eqs. (19) through (24). Thus stresses and displacements are either anti-symmetric (odd functions) or symmetric (even functions) with regard to \( z = 0 \). Orthogonality requires that Eigenvalues \( \alpha_n \) are chosen such that \( \cos(\alpha_n L) = 0 \) in Eq. (26). Accordingly, solutions fields with cosines or sines are zero at \( z = n L \), with \( n = 1, 3, 5 \ldots \) or \( n = 0, 2, 4, \ldots \), respectively, and their period is \( 4 L \). However, the stress and displacement behavior of the confined elastic medium is not periodic, but exponential and monotone decreasing with respect to \( z \) as indicated qualitatively by the Janssen model Eqs. (2) to (4). Despite the global periodicity of the trigonometric series, it is possible to obtain the desired exponential behavior locally within the sampling interval \([0; L]\). This is illustrated in Fig. 13, where the shear stress at the lateral boundary for the example problem is plotted beyond the sampling interval \( L = 200 \).

**Error Estimation**

Whereas the solution components \( S^{I} \) and \( S^{III} \) are given in closed form by Eqs. (14a-e) and (35a-e), respectively, the Fourier cosine transforms in \( S^{II} \) can be evaluated only computationally. Three different error sources appear during the evaluation of \( S^{II} \), namely the sampling error, the truncation error and the error resulting from numerical integration. The discussion here is focused on the former two, since the latter is reduced to a comparably very small size by dense discretization of the axisymmetric plane.

The sampling error occurs towards the end of the sampling interval \( z \in [0; L] \) and results from the fact that the series periodicity, which requires the cosine series to be already zero at
\( z = L \), is forced upon a monotone function, which only asymptotically tends towards zero for \( z \to \infty \) (Jerri 1992). The sampling error results in the deviation of the trigonometric stress and displacement fields from the expected monotone exponential behavior that culminates at the end of the sampling interval. This is illustrated in Fig. 14 by the wall shear stress \( \sigma^H_{rz} (R) \) with the sampling interval \([0; 200]\) and a series truncation of \( n = 50 \).

Provided that only the stress and displacement fields within the interval \( z \in [0; Z] \) are of interest, the sampling error can be considerably reduced, if the sampling length \( L \) is larger than the maximum depth of interest \( Z \), because the sampling error appears only at the end of the sampling length (see Fig. 14). For the present example, \( Z \) is chosen four times smaller than \( L \), where the sampling error is still negligibly small.

The truncation error is the difference between the truncated series representation and the exact series representation of Eqs. (19) through (24) that consists of an infinite number \( n \to \infty \) of terms (Jerri 1992). Computationally, the evaluation of the corresponding constants \( A_n \) and \( B_n \) by Eq. (30) can be accomplished only for a finite \( n \), so that the series expressions must be truncated after \( n \) terms. The truncation error leads to oscillations (Jerri 1992) that are distinctly visible in Fig. 14.

**Convergence Behavior of the Iterative Algorithm**

Within the scope of this paper, the discussion of the convergence is restricted to a qualitative examination on the basis of the Coulomb’s coefficient \( \mu \) Eq. (10) and the K-function Eq. (33) that are evaluated after each iteration for the example problem from the horizontal stresses \( \sigma^{I+H}_{rr} \), vertical stresses \( \sigma^{I+H}_{zz} \), and shear stresses \( \sigma^{I+H}_{rz} \) and illustrated in Figs. 14 and 15.

In Fig. 14, curve 0 corresponds to the initial \( \mu \)-value obtained from the first evaluation of the analytical solution procedure with Janssen’s constant \( K \)-value. During the first two iterations (curves 1 and 2), the algorithm shows a fast convergence against the exact Coulomb’s coefficient \( \mu = 0.36 \). The maximum deviation can be reduced with only 2 iterations by
around 98% (curve 2). However, curve 3 that shows the friction coefficient after 10 iterations has been hardly improved compared to curve 2. The corresponding K-functions shown in Fig. 15 behave accordingly showing no visible convergence subsequent to the second iteration step. This phenomenon is likely to be due to the sampling and truncation errors, whose influence predominates for small deviations of $\mu$ thus preventing further convergence.

From a mathematical point of view, the present iterative algorithm is a so-called fixed point iteration (Sueli and Mayers 2003). Its overall convergence is insured here, because the initial Janssen shear stress is close to the desired exact elastic solution, which has been shown in the comparison between Janssen and analytical solutions (Figs. 11 and 12).

**The special practical case of finite length**

It must be pointed out that with the help of the presented analytical methodology, stresses and displacements can only be obtained for a confined elastic cylinder of semi-infinite length. The situation most relevant to practical application, however, is a container of finite length $Z$ that is rigidly supported at the bottom end. The corresponding boundary condition can be written as $u_z(r, Z) = 0$ at some specific depth $z = Z$. The corresponding elastic solution fields cannot be obtained here, because vertical displacements can never be completely zero at any cross section due to the quadratic term of $u_z^{III}$ in Eqs. (31e). Under the assumption of a weightless material with $\rho = 0$ or in the special case of space applications, where gravity is zero, component $S_z^{III}$ is entirely zero. The remaining solution $S_z^I + S_z^{II}$ is able to satisfy $u_z(r, Z) = 0$ at $L = Z$ in Eq. (19) due to the orthogonality condition $\cos(\alpha L) = 0$. However, one has to deal with the sampling error at the end of the sampling interval $z \in [0; Z]$ in this case (see Fig. 14). Further comments on modeling of elastic stress and displacement profiles at the rigid bottom of a granular column can be found in Ovarlez and Clément (2005).
Summary and Conclusions

In this paper, an analytical solution methodology has been presented to obtain the complete stress and displacement fields for a laterally confined granular column idealized as a semi-infinite homogeneous elastic medium. The physical problem was translated into a mathematical boundary value problem that consisted of the governing equations of linear elasticity and the boundary conditions of constant top surcharge, lateral confinement and lateral Coulomb’s friction condition.

The corresponding solution methodology was then developed based on the superposition of three separate solution components. Whereas two of those, namely the components for the constant surcharge without wall friction and the weight of the material, could be given in closed form, the component for frictional shear at the wall was derived analytically by use of a potential trigonometric series approach with Bessel functions, finite Fourier transforms and the Janssen shear stress as a lateral boundary condition. The resulting solution fields analytically satisfied the governing equations and all boundary conditions except for the Coulomb’s friction condition at the lateral boundary. Therefore, the analytical solution procedure was embedded into an algorithm that iteratively approached the solution with exact Coulomb’s friction condition at the wall, while the governing equations and all other boundary conditions remained analytically satisfied. The present solution methodology was implemented into a computational MATLAB routine, and the elastic stress and displacement fields were computed for an example. The analytical results were found completely consistent with corresponding finite element computations and differed only marginally from the corresponding standard Janssen solution.

The major strength of the present solution methodology is its analytical nature that distinguishes it from purely numerical approaches such as the finite element method. Moreover, the axisymmetric solution provides both stresses and displacements at any point of the cylinder, whereas the one-dimensional Janssen model is limited to one specific value for axial, horizontal and shear stress at each depth $z$. A basic limitation of the present solution methodology is that it is not directly applicable to granular columns of finite lengths.
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References


function main
% Analytical solution methodology for a laterally confined elastic cylinder
global R L mu nu E N szzI alpha r_ele z_ele z_res z r

% 1. SPECIFY PARAMETERS

% Geometry
R=37.5;                % Radius
L=600;                 % Sampling length L

% Discretization
r_res = R/100:R/50:R-R/100;       % In r-direction
r_res(51) = R;
{r_ele = size(r_res,2);              % Number of elements in r-direction
z_res(1) = 0;                        % In z-direction
z_res(2:301) = L/600:L/300:L-L/600;
{z_ele = size(z_res,2);              % Number of elements in z-direction
[r,z] = meshgrid(r_res,z_res);       % 2D-Meshing

% Material
nu  = 0.3;          % Poisson's ratio
E   = 15;          % Young's modulus
gro = 0.0025;       % Mass density * acceleration due to gravity
mu  = 0.36;          % Coulob's friction coefficient
K   = zeros(1,z_ele);
K(:)= nu/(1-nu);           % Janssen’s K-value = elastic redirection coeff.
intK= K.*z_res;      % Pointwise integration to be used in Eq.(34)

P = 1;                  % Stress surcharge
P_red = P-(1-nu)*gro*R/(2*nu*mu);   % Reduction due to Eq.(36)
N = 200;                % Number of Fourier series elements
ite = 6;                % Number of iterations

% 2. COMPUTE SOLUTION COMPONENT S.I AND ORTHOGONALITY CONDITIONS

szzI = zeros(z_ele,r_ele);                % Solution component S.I Eqs.(14)
szzI(:, :) = P_red;                        % Vertical Stress
srrI = nu/(1-nu)*szzI;                    % Horizontal Stress
sthI = nu/(1-nu)*szzI;                    % Circumferential Stress
uzzI = szzI.*(1+nu)/E.*(2*nu-1)/(nu-1).*z;  % Vertical Displacement

for n=1:N
    alpha(n) = (2*n-1)/2*pi/L; % Orthogonality conditions Eq.(26)
end

% 3. ANALYTICAL SOLUTION PROCEDURE EMBEDDED IN THE ITERATIVE ALGORITHM

for i = 0:ite

    [A,B] = FourierTrans(K,intK);        % Finite Fourier cosine transforms

    srrII = zeros(1,z_ele);              % Assemble Horizontal Stresses

    for n=1:N
        srrII = srrII+alpha(n)^3.*sin(alpha(n)*z_res).*[A(n).* ...\n               (besseli(0, alpha(n)*R)-besseli(1, alpha(n)*R)./((alpha(n).*R)))+ ...\n               B(n).*((1-2*nu).*besseli(0, alpha(n)*R)+alpha(n).*R.* ...\n                  besseli(1, alpha(n)*R))];
    end

end
srr = srrI(:,r_ele)' + srrII; \ \% Superposition S.I + S.II
szzJ= P_red*exp(-2*mu/R*intK); \ \% Janssen vertical stress Eq.(1)
[K,intK] = K_eval(srr,szzJ); \ \% Update K(z) Eqs.(32)-(33)
end

[srrII,sthII,szzII,srzII,urrII,uzzII] = SeriesSum(A,B);

\% 4. COMPUTE S.III AND COMPLETE SUPERPOSITION
szzIII = zeros(z_ele,r_ele); \ \% Solution component S.III Eqs.(40)
szzIII(:, :) = (1-nu)*gro*R/(2*nu*mu); \ \% Vertical Stress
srrIII = nu/(1-nu)*szzIII; \ \% Horizontal Stress
sthIII = nu/(1-nu)*szzIII; \ \% Circumferential Stress
srzIII = 0.5*gro*r; \ \% Shear Stress
uzzIII = 1/E*((1+nu)/2*gro*r.^2+(1-nu-2*nu^2)*gro*R.*z/(2*nu*mu));

\% Superposition to complete analytical solution S.I + S.II + S.III
srr = srrI + srrII + srrIII;
sth = sthI + sthII + sthIII;
szz = szzI + szzII + szzIII;
srz = + srzII + srzIII;
urr = + urrII;
uzz = uzzI + uzzII + uzzIII;

\% Referential vertical displacements zero at (r,z)=(R,Z) (Eq.11)
\% Z is chosen here as L/2
u0 = zeros(z_ele,r_ele);
u0(:, :) = uzz(round(z_ele/4),r_ele);
uzz = uzz - u0;

\% 5. GRAPHICAL OUTPUT
figure(1); \ \% Convergence check
mu_eval = srz(:,r_ele)./srr(:,r_ele); \ \% Current mu close to exact mu?
plot(z_res,mu_eval),hold on

\% srr,sth,szz,urr,uzz can be plotted by replacing srz in the mesh-function
figure(2);
mesh(r,z,srz)
title('Semi-Analytical Solution: Shear Stress')
xlabel('Radius r'); ylabel('Depth from loading end z'); zlabel('Stress')
end

\% LIST OF FUNCTIONS
function [A,B] = FourierTrans(K,intK)
\% Finite Fourier cosine transforms
global R L mu nu N szzI alpha z_ele z_res
for n=1:N
B(n) = 0;
G = -2*alpha(n).^3*besseli(1, alpha(n)*R)*(-1+nu);
\% Compute B's from Eq.(30) by numerical integration and A's from Eq.(25)
for i=2:z_ele
B(n) = B(n)+1/G*2/L*cos(alpha(n)*z_res(i))*mu*K(i)*szzI(1,i)* ... 
exp(-2*mu/R*intK(i))*L/300;
end
A(n) = -B(n)*R*alpha(n)*besseli(0, alpha(n)*R)/besseli(1, alpha(n)*R);
end
end
function [srrII,sthII,szzII,srzII,urrII,uzzII]= SeriesSum(A,B)
% Fourier series summation
global nu E N alpha r_ele z_ele z r
srrII=zeros(z_ele,r_ele); sthII=zeros(zEle,r_ele);
srzII= zeros(z_ele,r_ele); szzII=zeros(z_ele,r_ele); urrII=zeros(z_ele,r_ele);
uzzII=zeros(z_ele,r_ele); for n=1:N
    srrII = srrII + alpha(n)^3.*sin(alpha(n)*z).* [A(n).* ... (besseli(0,alpha(n)*r)-besseli(1,alpha(n)*r)/(alpha(n).*r))+B(n).* ... ((1-2*nu).*besseli(0,alpha(n)*r)+alpha(n).*r.*besseli(1,alpha(n)*r))];
    sthII = sthII + alpha(n)^2.*r.*sin(alpha(n)*z).* [A(n).* ... besseli(1,alpha(n)*r)+B(n).* alpha(n).*r.* (1-2*nu).* ... besseli(0,alpha(n)*r)];
    szzII = szzII - alpha(n)^3.*sin(alpha(n)*z).* [A(n).* ... besseli(0,alpha(n)*r)+B(n).* (alpha(n).*r.*besseli(1,alpha(n)*r)+ ... 2.*(2-nu).*besseli(0,alpha(n)*r))];
    szzII = szzII - alpha(n)^3.*sin(alpha(n)*z).* [A(n).* ... besseli(0,alpha(n)*r)+B(n).* (alpha(n).*r.*besseli(1,alpha(n)*r)+ ... 2.*(2-nu).*besseli(0,alpha(n)*r))];
    urrII = urrII + (1+nu)/E.*sin(alpha(n)*z).* [A(n).* ... alpha(n).^2.*besseli(1,alpha(n)*r)+B(n).*alpha(n)^3.*r.* ... besseli(0,alpha(n)*r)];
    uzzII = uzzII + (1+nu)/E.*cos(alpha(n)*z).*alpha(n).^2.* [A(n).* ... besseli(0,alpha(n)*r)+B(n).* (4*(1-nu)*besseli(0,alpha(n)*r)+ ... alpha(n).*r.*besseli(1,alpha(n)*r))];
    end
end

function [K,intK] = K_eval(srr,szzJ)
% Evaluation of K(z) and its integration from 0 to each discrete point z
global R nu r_ele z_ele z_res
K = srr./szzJ;
% Pointwise integration from 0 to z to be used in Eq.(34)
fresult = fit(z_res',K','splineinterp'); % Spline interpolation
intK = integrate(fresult,z_res,0); % Analytical integration
end
**FIG. 1.** Elastic cylinder of semi-infinite length.

**FIG. 2.** Vertical equilibrium of a cylinder slice.
FIG. 3. Horizontal normal stress $\sigma_{rr}$: (a) analytical and (b) finite element results.
FIG. 4. Circumferential normal stress $\sigma_{\theta\theta}$: (a) analytical and (b) finite element results.
FIG. 5. Vertical normal stress $\sigma_{zz}$: (a) analytical and (b) finite element results.
FIG. 6. Shear stress $\sigma_{rz}$: (a) analytical and (b) finite element results.
FIG. 7. Horizontal displacements $u_r$: (a) analytical and (b) finite element results.
FIG. 8. Vertical displacements \( u_z \): (a) analytical and (b) finite element results.
FIG. 9. Coulomb’s friction coefficient $\mu$ in the completely superposed analytical solution (curve 1) and the finite element model (curve 2).

FIG. 10. Sketch of the finite element mesh and boundary conditions.
FIG. 11. Analytical shear stress at the lateral boundary (solid curves) and Janssen shear stress (dotted curves) for weight only (1), surcharge only (2) and weight + surcharge (3).

FIG. 12. Analytical averaged vertical stress (solid curves) and Janssen vertical stress (dotted curves) for weight only (1), surcharge only (2) and weight + surcharge (3).
FIG. 13. Shear stress at $r = R$ (dotted line = Janssen model, solid line = analytical results).

FIG. 14. Shear stress of component $S''$ at $r = R$ illustrating the sampling and truncation errors.
FIG. 15. Coulomb’s coefficients $\mu$ (curve 0 = initial, curve 1 = after 1 iteration, curve 2 = after 2 iterations, curve 3 = after 10 iterations).

FIG. 16. $K(z)$ -functions (curve 0 = initial elastic redirection coefficient, curve 1 = after 1 iteration, curve 2 = after 2 iterations, curve 3 = after 10 iterations).
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FIG. 12. Analytical averaged vertical stress (solid curves) and Janssen vertical stress (dotted curves) for weight only (1), surcharge only (2) and weight + surcharge (3).

FIG. 13. Shear stress at $r = R$ (dotted line = Janssen model, solid line = analytical results).

FIG. 14. Shear stress of component $S^n$ at $r = R$ illustrating the sampling and truncation errors.

FIG. 15. Coulomb’s coefficients $\mu$ (curve 0 = initial, curve 1 = after 1 iterations, curve 2 = after 2 iterations, curve 3 = after 10 iterations).

FIG. 16. $K(z)$-functions (curve 0 = initial elastic redirection coefficient, curve 1 = after 1 iteration, curve 2 = after 2 iterations, curve 3 = after 10 iterations).